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Suggested solutions

Exam: Introduction to Finance Mathematics (MT5009), 2022-08-17

Problem 1

(A) The value of the bond is $e^{-r}C + e^{-2r}(F + C) = 113.30$.
(B) The value of the bond is $e^{-5r} = 0.8607$. The value of the bond in $t \leq 5$ years is $e^{-r(5-t)}$. The second problem thus amounts to solving the equation $e^{-r(5-t)} = 0.9$ whose solution is

$$t = \frac{\ln(0.9)}{r} + 5 = 1.4880$$

which is therefore the answer.

Problem 2

Solution

(A) The spot rates are the yields $y(0, N)$ dictated by the current prices (see Capinski & Zastawniak, p. 247). We get the following equations for the yields

$$\begin{aligned} 95 &= 100e^{-y(0,1)}, \\ 92 &= 100e^{-2y(0,2)}. \end{aligned}$$

Hence

$$\begin{aligned} y(0,1) &= -\ln(95/100) \approx 5.13\%, \\ y(0,2) &= -\frac{1}{2}\ln(92/100) \approx 4.17\%. \end{aligned}$$

(B) If the term structure is deterministic, then the interest rate on a loan between time $t = 1$ and $t = 2$ will be given by the initial forward rate (see Capinski & Zastawniak, pp. 249-251),

$$f(0,1,2) = \frac{2y(0,2) - y(0,1)}{2-1} \approx 3.21\%.$$

(C)

- Time $t = 0$: You buy 100 one-year bonds for $100 \cdot 95$ SEK. To pay for this you borrow for two years at rate $y(0, 2)$ by shorting $9500/92 \approx 103.26$ two-year zero-coupon bonds.
- Time $t = 1$: You receive 10 000 SEK from the one-year bonds.
- Time $t = 2$: You close the short position in the two-year bonds, by paying $9500/92 \cdot 100 \approx 10\ 326.09$ SEK.

In this way you have constructed a loan between $t = 1$ and $t = 2$ at rate $\ln((9500/92 \cdot 100)/10000) = \ln(95/92) \approx 3.21\%$.

Problem 3

Solution

(A) Using the notation in Capinski & Zastawniak, we have

$$\mathbf{C} = \begin{pmatrix} \sigma_1^2 & c_{12} \\ c_{12} & \sigma_2^2 \end{pmatrix},$$

hence

$$\mathbf{C}^{-1} = \frac{1}{\sigma_1^2\sigma_2^2 - 2c_{12}} \begin{pmatrix} \sigma_2^2 & -c_{12} \\ -c_{12} & \sigma_1^2 \end{pmatrix},$$

hence

$$\mathbf{w}_{\text{MVP}} = \frac{\mathbf{u}\mathbf{C}^{-1}}{\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^\top} = \frac{1}{\sigma_1^2 + \sigma_2^2 - 2c_{12}} \begin{pmatrix} \sigma_2^2 - c_{12} & \sigma_1^2 - c_{12} \end{pmatrix}.$$

Using that $c_{12} = \rho_{12}\sigma_1\sigma_2$ we obtain

$$\mathbf{w}_{\text{MVP}} \approx \big(0.829, 0.171\big).$$

(B) The optimization problem is

$$\begin{aligned} &\text{minimize } \mathbf{w}\mathbf{C}\mathbf{w}^\top, \\ &\text{subject to } \mathbf{w}\mathbf{u}^\top = 1. \end{aligned}$$

(C) See Capinski & Zastawniak, p. 73.

Problem 4

Consider the following scheme:

- At $t = 0$ (today) lend $100/1.03^2$ USD for two years at 3%; this has cash flow 100 USD in two years.
- At $t = 0$ borrow $100/1.03^2$ EUR for two years at 1%; this has cash flow $-100 \cdot 1.01^2/1.03^2$ EUR in two years.

This scheme gives zero cash flow at $t = 0$ (since a EUR and a USD currently ave the same value) and the cash flow in EUR at $t = 2$ is

$$100(S(2) - 1.01^2/1.03^2)$$

in two years; where $S(2)$ is the (random) EUR value of a USD in two years. Under the no-arbitrage assumption we thus find that the forward price is

$$100 \cdot 1.01^2/1.03^2 = 96.15421.$$

Problem 5

We find a replicating portfolio according to

$$x(1 + K_{EUR})S^u + y(1 + K_{SEK}) = C^u = max\{S^u - X; 0\}$$
$$x(1 + K_{EUR})S^d + y(1 + K_{SEK}) = C^d = max\{S^d - X; 0\}$$

where x is the number of EUR in the replicating portfolio and y is the number of SEK. Plugging in the numbers and solving yields

$$x = 0.4902$$
$$y = -4.4554.$$

Since the value of the replicating portfolio must (by the no arbitrage principle) be equal to the price of the derivative we conclude that it is

$$C(0) = xS(0) + y = 0.4465.$$