Department of Mathematics
Stockholm University

MA 5020 - Abstract Algebra

Final Exam

Examiner: Gregory Arone
Date: August 12, 2022

- You may use the text (Dummit and Foote).
- You may **not** use class notes and/or any notes and study guides you have created.
- You may **not** use a calculator, a cell phone or computer.
- You may quote results that are proved in the book. When you do, state precisely the result that you are using, or give a precise pointer to the book.
- Be sure to justify your answers, and show clearly all steps of your solutions.
- In problems with multiple parts, results of earlier parts can be used in the solution of later parts, even if you do not solve the earlier parts

---

1. Let $H \subset S_4$ be the subgroup generated by $(1,3)$ and $(1,2,3,4)$.
   (a) (2 points) List the elements of H.
   (b) (2 points) Is H a normal subgroup of S_4?

2. Let G be a group with the property that for every $x \in G$, $x^2 = e$
   (a) (2 points) Prove that G is abelian.
   (b) (2 points) Suppose that G is also finite. Prove that the number of elements of G is a power of 2.

3. Suppose G is a group acting on a set X. Recall that the action is said to be
   - *transitive* if for all $u, v \in X$, there exists a $g \in G$ such that $gu = v$.
   - *free* if for all $g \in G \setminus \{e\}$ and all $x \in X, gx \neq x$.

   Suppose K and H are subgroups of G. Let G/H denote the set of left cosets of H. Then K acts on G/H by the formula $k \cdot (gH) = (kg)H$. This is the restriction of the standard action of G on G/H.
   (a) (2 points) Prove that the action of K on G/H is transitive if and only if $KH = G$.
   (b) (2 points) For which values of n is the action of A_n on S_n/C_n transitive? Here A_n denotes the alternating group, and C_n is the cyclic subgroup of S_n generated by the cycle $(1, 2, ..., n)$.
   (c) (3 points) Let p and q be distinct primes. Suppose that P and Q are a p-subgroup and a q-subgroup of G respectively. Prove that the action of P on G/Q is free.

4. (a) (3 points) Prove that a group with 132 elements can not be simple.
   (b) (3 points) Prove that a group with 216 elements can not be simple.

5. (3 points) Find all the maximal ideals of the ring $\mathbb{Z} \times \mathbb{Z}$.

   Hint: show that every ideal of $\mathbb{Z} \times \mathbb{Z}$ is of the from $I \times J$, where I and J are ideals of $\mathbb{Z}$.

6. Let $R = \mathbb{Z}[\sqrt{-5}]$ be the subring of $\mathbb{C}$ consisting of elements of the form $a + b\sqrt{-5}$, where a and b are integers. Let I be the ideal of R generated by 2 and $1 + \sqrt{-5}$. We can write $I = (2, 1 + \sqrt{-5})$. Similarly, let $J = (3, 2 - \sqrt{-5})$.

(a) (3 points) Prove that I is not a principal ideal.
Remark: it is also true that J is not principal, but you are not required to show that.

(b) (3 points) Prove that $IJ = (1 + \sqrt{-5})$. In particular, IJ is principal.