MM8019 Algebraic Geometry and Commutative Algebra
Final exam
January 8, 2020

Time: 09:00-14:00
Aids: none
Examinor: Dan Petersen

The final grade will be based on the sum of the credits from the homework assignments and the result on the final exam and will preliminarily be given by the following table, given that at least 50 points comes from the final exam:

Grade	A	B	C	D	E	Fx	F
Points	200-240	180-199	160-179	140-159	120-139	110-119	0-109

1. (a) Define the following concepts: *finite module* and *split exact sequence*. **(5 p)**
   (b) Show that $M_3 \cong M_2/\operatorname{im}(\partial_1)$ if
   $$0 \longrightarrow M_1 \xrightarrow{\partial_1} M_2 \xrightarrow{\partial_2} M_3 \longrightarrow 0$$
   is a short exact sequence of modules. **(7 p)**
   (c) Formulate *Nakayama's Lemma* and prove it by induction over the number of generators of the finite module. **(8 p)**

2. (a) Define the following concepts: *integral extension*, *normal domain*. **(5 p)**
   (b) Let $R = k[x, y]$ where k is a field with $\operatorname{char}(k) \neq 2$, let $z_1 = x + y$ and let $z_2 = x^2 + y^2$. Show that z_1 and z_2 are algebraically independent and that R is finite over the subring $S = k[z_1, z_2]$. (Hint: Show that x and y are integral over S.) **(15 p)**

3. (a) Define the following concepts: the *coordinate ring* of a variety V and the *radical* of an ideal I. **(5 p)**
   (b) Let k be a field. A *monomial ideal* is an ideal in a polynomial ring $k[x_1, x_2, \dots, x_n]$ generated by *monomials*, i.e., by elements of the form $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where $\alpha_i \in \mathbb{N}$, for $i = 1, 2, \dots, n$. A monomial ideal is *square-free* if it is generated by *square-free* monomials, i.e., monomials where $\alpha_i \leq 1$ for all $i = 1, 2, \dots, n$. Prove that a monomial ideal is radical if and only if it is square-free. **(10 p)**
   (c) Find the irreducible components of the monomial ideal
   $$I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1) \subseteq k[x_1, x_2, \dots, x_5],$$
   where k is a field. **(5 p)**

4. (a) Let S be a multiplicative set in a commutative ring A. There is a simple description of the set of prime ideals in $S^{-1}A$ in terms of the prime ideals of A and how they meet the set S — state this description. **(4 p)**
   (b) Describe all possible localizations of the ring $\mathbb{Z}/24\mathbb{Z}$. Justify your answer. **(8 p)**
   (c) Give an example of an element which is not in the image of the natural map
   $$\mathbb{Z}[[x]] \otimes_{\mathbb{Z}} \mathbb{Z}[[y]] \to \mathbb{Z}[[x, y]].$$
   (8 p)

5. (a) Define what it means for an ideal in $k[x_0, \ldots, x_n]$ to be *homogeneous*. **(4 p)**
   (b) Consider the affine curve $C_f \subset \mathbb{A}^2_{\mathbb{C}}$ defined by the equation $y^2 = f(x)$, where $f \in \mathbb{C}[x]$ is a polynomial. Show that C_f is smooth if and only if f is squarefree. **(8 p)**
   (c) Suppose that $F(X, Y) = 0$ is a plane curve in $\mathbb{A}^2_{\mathbb{C}}$, and that the origin is a singular point of F. Show that the origin is a simple node (a.k.a. an ordinary double point) of the curve if and only if
   $$\frac{\partial^2 F}{\partial X^2}(0,0) \cdot \frac{\partial^2 F}{\partial Y^2}(0,0) \neq \left(\frac{\partial^2 F}{\partial X \partial Y}(0,0) \right)^2.$$
   (Hint: a polynomial $ax^2 + bx + c$ is a perfect square if and only if $b^2 - 4ac = 0$.)
   (8 p)

6. (a) Define the *tangent line* to a curve at a smooth point. **(4 p)**
   (b) Let $F(x, y) = x^4 + x^3y + y^2$ and $G(x, y) = x + y$. Consider the two affine plane curves $F(x, y) = 0$ and $G(x, y) = 0$ over $\mathbb{C}$. Find all intersection points between F and G, and find the intersection multiplicities. Then do the same thing for the homogenizations of these polynomials, and the corresponding curves in $\mathbb{P}^2$. **(8 p)**
   (c) Consider an irreducible degree four projective plane curve, with a singular point of multiplicity three. Show that it has no further singular points. (Hint: use Bézout.) **(8 p)**