Solutions for the exam 24/10

Exercise 1

$f(x) = x\,e^{x^2}$ $\qquad f(0) = 0$ $\quad$ 0.5

$f'(x) = e^{x^2} + 2x^2 e^{x^2}$ $\quad$ 0.5 $\qquad f'(0) = 1$ $\quad$ 0.5

$f''(x) = 2xe^{x^2} + 4x\,e^{x^2} + 4x^3 e^{x^2}$ $\quad$ 0.5 $\qquad f''(0) = 0$ $\quad$ 0.5

$f'''(x) = 2e^{x^2} + 4x^2 e^{x^2} + 4e^{x^2} + 8x^2 e^{x^2} + 12x^2 e^{x^2} + 8x^4 e^{x^2}$ $\quad$ 0.5

$f'''(0) = 6$ $\quad$ 0.5

$$p(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{6}(x-0)^3$$

$$= 0 + 1\cdot x + \frac{0}{2}\cdot x^2 + \frac{6}{6}x^3$$

$$= \boxed{x + x^3}$$ 0.5

$p(0.1) = \underset{0.5}{0.1 + (0.1)^3} = \boxed{0.101}$ $\quad$ 0.5

Exercise 2

(a) We first need to write

$$S(x) = \sum a\,q(x)^n$$

a = first term of the sum = 5 1 pt

$q(x) = \text{2nd term}/5 = \frac{-10e^{-x}}{5} = -2e^{-x}$ 1 pt

For convergence we need

$$|-2e^{-x}| < 1$$ 0.5 pt

$\Leftrightarrow \quad 2e^{-x} < 1$

$\Leftrightarrow \quad 2 < e^{x} \quad \Leftrightarrow \quad \boxed{\ln(2) < x}$ 0.5 pt

(b) We have that, for $x > \ln(2)$,

$$S(x) = \frac{a}{1-q(x)} = \frac{5}{1+2e^{-x}}$$

thus we need to solve

$$\frac{5}{1+2e^{-x}} = \frac{1}{3}$$ 1 pt for correct setting up.

$\Leftrightarrow \quad 15 = 1+2e^{-x}$

$14 = 2e^{-x}$

$7 = e^{-x} \quad \Leftrightarrow \quad \ln(7) = -x$ 0.5

$x = -\ln(7) < \ln(2)$

For this x $S(x)$ does not converge so there are no x such that $S(x) = \frac{1}{3}$ 0.5

Exercise 3

(a) $f(x) = (2x+1)e^{-x^2+1}$

$$f'(x) = 2e^{-x^2+1} - 2x(2x+1)e^{-x^2+1}$$

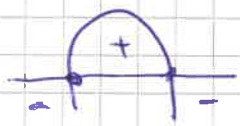
$$= e^{-x^2+1}(-4x^2-2x+2)$$ (0.5 pt)

We want to study when $f'(x) \geq 0$. To do that we find the roots of $-4x^2-2x+2$

$$x_\pm = \frac{+2 \pm \sqrt{(-2)^2-4(-4)\cdot 2}}{-8} = \frac{2 \pm \sqrt{36}}{-8}$$

$$= \begin{cases} \frac{8}{-8} = -1 \\ \frac{-4}{-8} = +\frac{1}{2} \end{cases}$$ (0.5 pt)

As the coefficient of x^2 is negative we have that it is a parabola like that



It is positive between the roots and negative outside the roots

		-1		$1/2$	
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	$\to$	$\nearrow$	$\to$	$\searrow$

(1 pt)

So (-1) is a local minimum & $\frac{1}{2}$ is a local max

Alternatively: After finding $x_\pm$ one compute

$$f''(x) = (-8x-2)e^{-x^2+1} - 2x(-4x^2-2x+2)e^{-x^2+1}$$

$$= (+8x^3+4x^2-8x)e^{-x^2+1}$$

(1 pt).

Compute $f''(-1) = \cancel{-8}+4+\cancel{8} > 0$ local min

$f''(\frac{1}{2}) = (1+1-4)e^{3/4} < 0$ local max

(b) From (a) we have that f is increasing when $x \in [-1, \frac{1}{2}]$ and decreasing when $x \in (-\infty, -1] \cup (-\frac{1}{2}, +\infty)$

Points

If the sign was studied in (a) 1 point for the consistence. Otherwise 0.5 pt for sign study + 0.5 for conclusion.

(d) $\lim_{x\to+\infty} (2x+1)e^{-x^2+1} = [+\infty \cdot 0]$ indetermined.

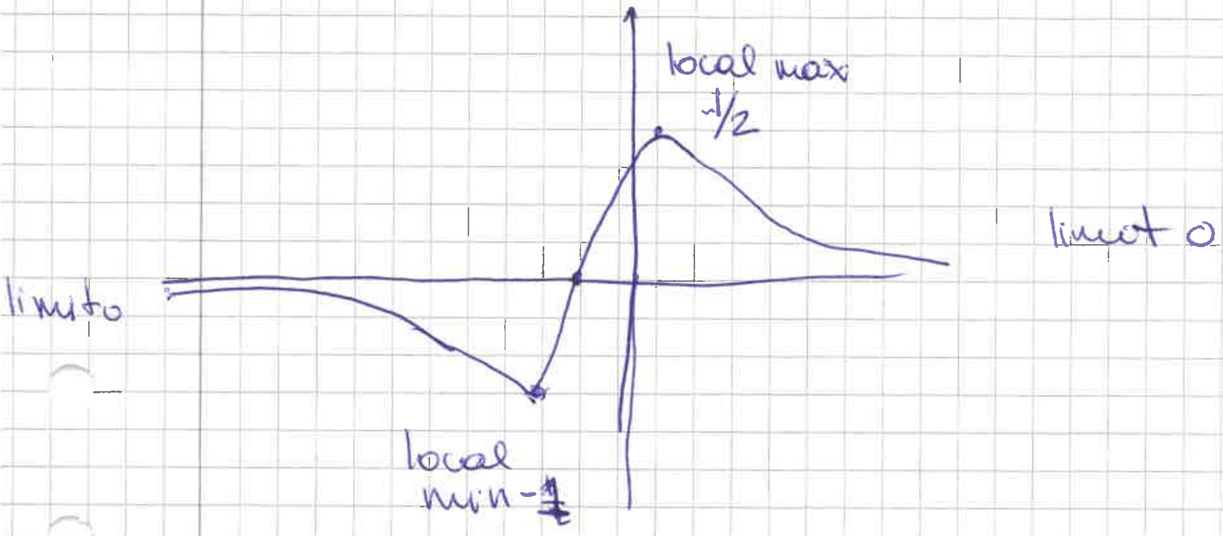
$= \lim_{x\to+\infty} \frac{(2x+1)}{e^{x^2-1}} = \left[\frac{+\infty}{+\infty}\right]$ we can use L'Hôpital

$= \lim_{x\to+\infty} \frac{2}{2xe^{x^2-1}} = \left[\frac{2}{+\infty}\right] = 0$

With $-\infty$ it is the same

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2x+1}{e^{x^2-1}} \overset{H}{=} \lim_{x\to-\infty} \frac{2}{2xe^{x^2-1}} = \left[\frac{2}{-\infty}\right] = 0$

For the graph
we observe $f(x)=0$ only if $x=-\frac{1}{2}$



Points 0.5 for the limit It is enough to observe that e wins over $2x+1$ as we did in class
0.5 for the graph.

(c) Of the critical pts that we found only $1/2$ is in the interior of the interval. We evaluate f at $1/2$ & in the extrema

$f(1/2) = 2e^{3/4} \approx 4.234$

$f(-1) = -e^{-1+1} = -1$

$f(2) = 5e^{-3} = 0.25$

we have that the max vale is $2e^{3/4}$ & the min value is -1

Points 0.5 for individuating the right points to evaluate
0.5 for conclusion

Exercise 4

(a) $\int \left(\sqrt{t}\, e^{\sqrt{t}} + \sqrt[5]{t^3} \right) dt =$

$$= \int \sqrt{t}\, e^{\sqrt{t}}\, dt + \int \sqrt[5]{t^3}\, dt =$$

$u = \sqrt{t}$

$du = \frac{1}{2\sqrt{t}}\, dt$

$dt = 2u\, du$

$$= \int u e^{u}\, 2u\, du + \int t^{\frac{3}{5}}\, dt$$

$$= 2\left[u^2 e^u - \int 2u e^u\, du \right] + \frac{1}{1+\frac{3}{5}}\, t^{3/5+1} + C_1$$

$$= 2\left[u^2 e^u - 2ue^u + 2\int e^u\, du \right] + \frac{5}{8}\, t^{8/5} + C_1$$

$$= 2u^2 e^u - 4ue^u + 4e^u + C_2 + \frac{5}{8}\, t^{3/5} + C_1$$

$$= 2t e^{\sqrt{t}} - 4\sqrt{t}\, e^{\sqrt{t}} + 4e^{\sqrt{t}} + \frac{5}{8}\, t^{3/5} + C$$

Points

0.5 for splitting the integral
0.5 for 2nd integral correct
0.5 for having C
0.5 for correct variable sub
0.5 for 1st integration by parts
0.5 for 2nd integration by parts

(b)

$$\int_0^1 \frac{3y}{y^2+1}\,dy$$

$u = y^2+1$
$du = 2y\,dy$
$y=0 \quad u=1$
$y=1 \quad u=2$

$$\frac{3}{2}\int_1^2 \frac{du}{u} = \frac{3}{2}\Big[\ln u\Big]_1^2 = \frac{3}{2}\left(\ln(2)-\ln(1)\right)$$

$$= \frac{3}{2}\ln(2)$$

Points: 0.5 for the change of variable
0.5 for the extremes
0.5 for the primitive
0.5 for the final result.

Exercise 5

a) $\det(A) = (-1)^{2+2}\cdot(-1)\det\begin{pmatrix} 2 & 2+c \\ c & -2 \end{pmatrix} =$

$= -1\left(-4 - c(2+c)\right) =$

$= -1\left(-4 - 2c - c^2\right)$

$= \left(c^2+2c+4\right)$

2 points for correct formula
1 pt for correct executi

1 point.

b) we need to solve $c^2+2c+4=0$

0.5

$\Delta = 4-16 < 0$ this has no solution

So this ~~has~~ is always invertible. There is no value of c for which A is not invertible 0.5

(c)

We find the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 0 & -2 & 4 \\ 3 & -1 & 3 & 13 \\ -2 & 0 & -2 & -8 \end{array}\right) \quad R_2 \longleftrightarrow R_1$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 3 & 13 \\ 2 & 0 & -2 & 4 \\ -2 & 0 & -2 & -8 \end{array}\right) \quad R_3 \longrightarrow R_3 + R_2$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 3 & 13 \\ 2 & 0 & -2 & 4 \\ 0 & 0 & -4 & -4 \end{array}\right)$$

1 pt for Gauss E.

the third equation is $-4z = -4 \Rightarrow \boxed{z = 1}$

the second equation is $2x - 2z = 4$

we plug in $z = 1$ and get $2x - 2 \cdot 1 = 4$

$\Leftrightarrow 2x = 6 \quad \boxed{x = 3}$

the first equation is

$3x - y + 3z = 13$

$9 - y + 3 = 13$

$-y = 13 - 12$

$\boxed{y = -1}$

1 point for solution at the end.

The solution is $(3, -1, 1)$

Exercise 6

b) (a)
$$\begin{cases} \partial_x f(x,y) = (y-1)\, e^{xy-x-y} = 0 \\ \partial_y f(x,y) = (x-1)\, e^{xy-x-y} = 0 \end{cases}$$

$x = 1$
$y = 1$ 1 pt we have only one critical point $(1,1)$

We compute the Hessian

$$\partial_{xx} f(x,y) = (y-1)^2 e^{xy-x-y}$$
$$\partial_{yy} f(x,y) = (x-1)^2 e^{xy-x-y}$$
$$\partial_{yx} f(x,y) = e^{xy-x-y} + (y-1)(x-1) e^{xy-x-y}$$
$$\partial_{xy} f(x,y) = e^{xy-x-y} + (x-1)(y-1) e^{xy-x-y}$$

$$\det H = (x-1)^2(y-1)^2 e^{2(xy-x-y)} - \left(e^{xy-x-y} + (x-1)(y-1)e^{xy-x-y}\right)^2$$

$$= \cancel{(x-1)^2(y-1)^2 e^{2(xy-x-y)}} - e^{2(xy-x-y)} - 2(x-1)(y-1)e^{2(xy-x-y)} - \cancel{(x-1)^2(y-1)^2 e^{2(xy-x-y)}}$$

$$= -e^{2(xy-x-y)} - 2(x-1)(y-1)e^{2(xy-x-y)}$$

We compute in $(1,1)$ and get

$$-e^{2(1-1-1)} = -e^{-2} < 0$$

~~we checked the value of ∂_{xx} and ∂_{yy} in $(1,1)$~~

~~and we~~ We have a saddle point ✓

1 pt.

(b) The border has 3 corners
(0,0) (0,4) and (4,0)

0.5 for the corners

The 3 sides are

$\{x=0,\ y\in[0,4]\}$

$\{y=0,\ x\in[0,4]\}$

$\{y=4-x\ \ x\in[0,4]\}$

0.5 for the candidates in each side.

Side 1 $x=0$

$g(y)=f(0,y)=e^{-y}$

$g'(y)=-e^{-y}\neq 0$ There are no candidates on this side

Side 2 $y=0$

$g(x)=f(x,0)=e^{-x}$

$g'(x)=-e^{-x}\neq 0$ There are no candidates on this side

Side 3 $y=4-x$

$$g(x)=f(x,4-x)=e^{x(4-x)-(4-x)-x}=$$

$$=e^{4x-x^2-4+\cancel{x}-\cancel{x}}$$

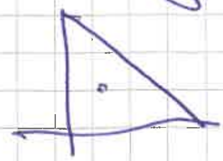
$$=e^{-x^2+4x-4}$$

$g'(x)=(-2x+4)e^{-x^2+4x-4}=0$ iff $x=2$

$\Rightarrow y=4-2=2$

we get that (2,2) is a candidate for max or min

(c) ~~(i)~~ From (a) & (b) we get the following candidates ∅ evaluation



(0 0)	$f(0,0) = 1$
(4 0)	$f(4\ 0) = e^{-4}$
(0 4)	$f(0\ 4)\ e^{-4}$
(1,1)	$f(1\ 1) = e^{-1}$
(2 2)	$f(2\ 2) = 1$

⚠ you need to check that (1,1) is inside D

The max value is 1, The min value is e^{-4}

Points

0.5 for deducing the correct list given your solution in (a) & (b)

0.5 for the conclusion