Exam Probability theory III October 24, 2022

Part I

1 A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ corresponding to the experiment can be constructed as follows:

$$\Omega = \{HH, HT, TH, TT\}$$
$$\mathcal{F} = \text{power set of } \Omega$$
$$\mathbb{P}(\{\omega\}) = \frac{1}{4} \quad \text{for each } \omega \in \Omega.$$

One may easily verify that the choice of $\mathbb{P}$ corresponds to the outcomes of each toss having equal probability. 3p

We note that the collection

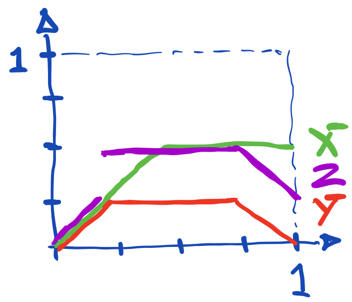
$$\mathcal{A} = \{\underbrace{\{\text{two heads}\}}^{A_2}, \underbrace{\{\text{one heads}\}}^{A_1}, \underbrace{\{\text{no heads}\}}^{A_0}\}$$

forms a partition of Ω. The σ-algebra generated by $\mathcal{A}$ consists of all subsets of Ω that can be formed taking unions of events in $\mathcal{A}$. There are $2^{|\mathcal{A}|} = 8$ such sets.

$$\sigma(\mathcal{A}) = \{\emptyset, A_0, A_1, A_2, A_0 \cup A_1, A_0 \cup A_2, A_1 \cup A_2, A_0 \cup A_1 \cup A_2\}.$$

2p

2 For visualization we make a drawing of the three random variables.



From the definition we see that

$$\{X=Y\} = [0, 1/4]$$
$$\{Y=Z\} = [0, 1/4)$$
$$\{X=Z\} = [0, 1/4) \cup [1/2, 3/4]$$

2p

Since $\mathbb{P}$ denotes Lebesgue measure, the probability of these sets is simply their lengths. Hence

$$\mathbb{P}(X=Y) = 1/4 \qquad \mathbb{P}(Y=Z) = 1/4 \qquad \mathbb{P}(X=Z) = 1/2.$$

3p

additivity used here

$\boxed{3}$ Let $A_n = \{X_n \geq n\}$. We want to prove that

$$\mathbb{P}(A_n \text{ occurs i.o.}) = 0.$$

Using Chebyshev's inequality we find that

$$\mathbb{P}(X_n \geq n) \leq \mathbb{P}(|X_n| \geq n) \leq \frac{Var(X_n)}{n^2} = \frac{1}{n^2}.$$ 2p

It follows that

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

The first Borel-Cantelli lemma implies that

$$\mathbb{P}(A_n \text{ i.o.}) = 0.$$ 3p

$\boxed{4}$ Since S_n is a function of $X_1, \dots, X_n$ it is clearly $\mathcal{F}_n$-measurable. Moreover, 1p

$$\mathbb{E}[|S_n|] \leq \sum_{k=1}^{n} c^k \cdot \mathbb{E}[|X_k|]$$

which is finite for all $n \geq 1$ since the variables $X_1, \dots, X_n$ are integrable by assumption that their means are well-defined. 1p

Finally we verify the martingale property:

$$\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[S_n + c^{n+1} \cdot X_{n+1} \mid \mathcal{F}_n]$$

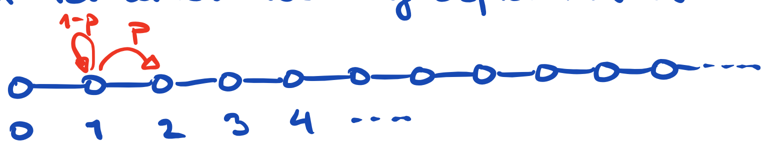
linearity, S_n is $\mathcal{F}_n$-measurable $\rightarrow$ $= S_n + c^{n+1} \cdot \mathbb{E}[X_{n+1} \mid \mathcal{F}_n]$ 3p

$= S_n$

independence

Part II

5 One may recognise $(S_n)_{n\geq 0}$ as a random walk on the positive integers which in each step jumps one step up or remains. We may depict this as



(a) We have $S_0 = 0$ and $S_n = \sum_{k=1}^{n} X_k$, and

$$T = \min\{n \geq 1 : S_n \geq a - n\}$$

where $a > 0$ is an integer. Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

We have

$$\{T = n\} = \{S_1 < a-1\} \cap \{S_2 < a-2\} \cap \ldots \cap \{S_{n-1} < a-(n-1)\} \cap \{S_n \geq a-n\}.$$

Since S_n is a function of $X_1, \ldots, X_n$ it is $\mathcal{F}_n$-measurable. Hence $\{T=n\}$ is the intersection of elements in $\mathcal{F}_n$, and thus in $\mathcal{F}_n$ itself. So T is a stopping time wrt $(\mathcal{F}_n)_{n\geq 1}$. 8p

(b) We note that $S_n \geq 0$ for all $n \geq 0$, so $T \leq a$ and hence bounded. We also note that

$$\mathbb{E}[X_k] = \mathbb{P}(X_k = 1) = p. \qquad 2p$$

Set $Y_n = S_n - pn$. Clearly Y_n is $\mathcal{F}_n$-measurable and integrable, and

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = Y_n + \mathbb{E}[X_{n+1} - p \mid \mathcal{F}_n] = Y_n.$$

That is, $(Y_n)_{n\geq 0}$ is a martingale wrt $(\mathcal{F}_n)_{n\geq 1}$. 2p

By the optional stopping thm I we have

$$\mathbb{E}[Y_T] = \mathbb{E}[Y_0] = 0.$$

It follows that

$$p\mathbb{E}[T] = \mathbb{E}[S_T] \geq a - \mathbb{E}[T].$$

Hence 3p

$$\mathbb{E}[T] \geq \frac{a}{1+p}.$$

$\boxed{6}$ (a) Since $\mathbb{P}(|X_n| \le K) = 1$ for all $n \ge 1$ and

$$\mathbb{P}(|X_n - X| > \varepsilon) \to 0 \qquad \text{as } n \to \infty$$

it follows that

$$\mathbb{P}(|X| > K+1) \le \mathbb{P}(|X - X_n| > 1) + \mathbb{P}(|X_n| > K).$$

The RHS tends to zero as $n \to \infty$, so the LHS $= 0$. 3p

(b) Let $A_n = \{|X_n - X| \le \varepsilon\}$. Then

$$\mathbb{E}[|X_n - X|] = \mathbb{E}[|X_n - X| \mathbb{1}_{A_n}] + \mathbb{E}[|X_n - X| \mathbb{1}_{A_n^c}]$$
$$\le \varepsilon + (2K+1) \cdot \mathbb{P}(A_n^c)$$
$$\le 2\varepsilon$$

for large values of n. Since $\varepsilon > 0$ was arbitrary, it follows that

$$\mathbb{E}[|X_n - X|] \to 0 \qquad \text{as } n \to \infty.$$ 4p

(c) Finally, as $n \to \infty$, using Jensen's inequality

$$|\mathbb{E}[X] - \mathbb{E}[X_n]| \le \mathbb{E}[|X - X_n|] \to 0.$$ 3p

$\boxed{7}$ (a) Any function $X : \Omega \to \mathbb{R}$ is a random variable as $\mathcal{F}$ contains all subsets of Ω, so

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$$

is trivially contained in $\mathcal{F}$. 3p

(b) The number of elements in Ω is 2^n, and out of these half of them have k^{th} coordinate equal to 1. Using (countable) additivity we find that

$$\mathbb{P}(X_k = 1) = \sum_{\substack{\omega \in \Omega \\ \omega_k = 1}} \mathbb{P}(\{\omega\}) = 2^{n-1} \cdot \frac{1}{2^n} = \frac{1}{2}.$$ 3p

(c) By observing X_k we can tell the outcome of the k^{th} coordinate of ω. So X_k will tell us which of the two events

$$A_k = \{\omega \in \Omega : \omega_k = 1\} \quad \text{and} \quad A_k^c = \{\omega \in \Omega : \omega_k = 0\}$$

that has occured. $\sigma(X_k)$ is the σ-algebra generated by the partition $\mathcal{P} = \{A_k, A_k^c\}$.

Alternatively, note that for Borel sets $B \subseteq \mathbb{R}$

$$X_k^{-1}(B) = \begin{cases} \emptyset & \text{if} \quad 0,1 \notin B \\ A_k^c & \text{if} \quad 0 \in B, 1 \notin B \\ A_k & \text{if} \quad 0 \notin B, 1 \in B \\ \Omega & \text{if} \quad 0,1 \in B \end{cases}$$

4p

This is precisely the σ-algebra generated by $\mathcal{P}$.

$\boxed{8}$ (a) Let $X_1, X_2, \ldots$ be defined as

$$X_k = \begin{cases} 1 & \text{ball drawn in round } k \text{ is red,} \\ 0 & \text{otherwise.} \end{cases}$$

Then we may write

$$Y_n = \frac{1}{n+3}\left[1 + \sum_{k=1}^{n} X_k\right].$$

It is clear that Y_n is bounded, so therefore integrable.

Let $\mathcal{F}_n = \sigma(Y_1, Y_2, \ldots, Y_n)$. Then

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \frac{1}{n+4}\,\mathbb{E}\left[1 + \sum_{k=1}^{n+1} X_k \,\middle|\, \mathcal{F}_n\right]$$

$$= \frac{1}{n+4}\left[1 + \sum_{k=1}^{n} X_k + \mathbb{E}[X_{n+1} \mid \mathcal{F}_n]\right]$$

Given $\mathcal{F}_n$, the probability of the $n+1$st draw resulting in a red ball is Y_n. Hence

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \frac{1}{n+4}\left[(n+3)\,Y_n + Y_n\right] = Y_n.$$

3p

That is, $(Y_n)_{n\geq 0}$ is a martingale wrt $(\mathcal{F}_n)_{n\geq 1}$.

(b) $(Y_n)_{n\geq 0}$ is a bounded martingale, and hence almost surely convergent. By the bounded convergence theorem we have

$$\mathbb{E}[\lim_{n\to\infty} Y_n] = \lim_{n\to\infty} \mathbb{E}[Y_n] = \mathbb{E}[Y_0] = 1/3,$$

2p

since for martingales $\mathbb{E}[Y_n] = \mathbb{E}[Y_0]$.

(c) It will suffice to prove that the red ball initially present will be drawn infinitely many times. If this happens with probability 1, then all three balls will be drawn infinitely many times with probability one

Let $A_n = \{$ the original red ball drawn in round $n\}$. Since there are $n+2$ balls present when the nth draw is performed, the probability that the original red is the ball drawn is $\frac{1}{n+2}$. Thus

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \sum_{n=1}^{\infty} \frac{1}{n+2} = \infty.$$

Moreover, since different draws are independent, the events $A_1, A_2, \ldots$ are independent. The second Borel-Cantelli lemma implies that $\mathbb{P}(A_n \text{ i.o.}) = 1$. 3p

(d) The time T when the 10^{th} red ball is added to the urn is a stopping time. Let $R_n = 1 + \sum_{k=1}^{n} X_k$ denote the number of red in the urn after n rounds. Then $R_n = (n+3) Y_n$ which is $\mathcal{F}_n$-measurable. Hence

$$\{T=n\} = \{R_1 < 10\} \cap \ldots \cap \{R_{n-1} < 10\} \cap \{R_n = 10\}$$

which are events in $\mathcal{F}_n$.

The sequence $(Z_n)_{n \geq 0}$ where $Z_n = Y_{n \wedge T}$ is again a martingale, and bounded. The bounded convergence thm gives 2p

$$\mathbb{E}[Y_T] = \mathbb{E}[\lim_{n\to\infty} Z_n] = \lim_{n\to\infty} \mathbb{E}[Z_n] = \mathbb{E}[Y_0] = 1/3.$$