MT7001 – Probability theory III – exam

Date Monday October 24, 2022
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Tools None.
Grading criteria The exam consists of two parts, which consist of 20 and 40 points respectively. To pass the exam a score of 14 or higher is required on Part I. If attained, then also Part II is graded, and the score on this part determines the grade. Grades are determined according to the following table:

	A	B	C	D	E
Part I	14	14	14	14	14
Part II	32	24	16	8	0

Problems of Part I may give up to five points each, and problems of Part II may give up to ten points each. Complete and well motivated solutions are required for full score. Partial solution may be rewarded with a partial score.

Part I

Problem 1. Consider the experiment of tossing a fair coin twice. Construct a probability space corresponding to this experiment. Determine the σ-algebra generated by the events that the experiment results in two, one or no 'heads'.

Problem 2. Consider the probability space $([0,1],\mathcal{B},\mathbb{P})$, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,1]$ and $\mathbb{P}$ denotes Lebesgue measure. Determine $\mathbb{P}(X=Y)$, $\mathbb{P}(Y=Z)$ and $\mathbb{P}(X=Z)$, where X, Y and Z are defined as

$$X(\omega)=\begin{cases}\omega & \text{for } \omega\in[0,1/2),\\ 1/2 & \text{for } \omega\in[1/2,1];\end{cases}$$
$$Y(\omega)=\begin{cases}\omega & \text{for } \omega\in[0,1/4),\\ 1/4 & \text{for } \omega\in[1/4,3/4),\\ 1-\omega & \text{for } \omega\in[3/4,1];\end{cases}$$
$$Z(\omega)=\begin{cases}\omega & \text{for } \omega\in[0,1/4),\\ 1/2 & \text{for } \omega\in[1/4,3/4),\\ 5/4-\omega & \text{for } \omega\in[3/4,1].\end{cases}$$

Problem 3. Let $X_1,X_2,\dots$ be random variables with mean zero and variance 1, defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$. Prove that the probability that "$X_n\geq n$ for infinitely many $n\geq 1$" is zero.

Problem 4. Let $X_1,X_2,\dots$ be i.i.d. random variables with mean zero and variance 1, and let $c\in(0,1)$ be a real number. Set $S_n=\sum_{k=1}^n c^kX_k$ and $\mathcal{F}_n=\sigma(X_1,\dots,X_n)$. Show that $(S_n)_{n\geq1}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq1}$.

Part II

Problem 5. Let $X_1, X_2, \ldots$ be independent random variables such that $\mathbb{P}(X_k = 1) = 1 - \mathbb{P}(X_k = 0) = p$, where $p \in (0,1)$. Set $S_0 = 0$ and $S_n = \sum_{k=1}^n X_k$. Let $a > 0$ be an integer and set $T = \min\{n \geq 1 : S_n \geq a - n\}$.

(a) Show that T is a stopping time.

(b) Prove that $\mathbb{E}[T] \geq a/(1+p)$.

Problem 6. Let $(X_n)_{n\geq 1}$ be a sequence of random variables that converges in probability to some random variable X, and suppose that $\mathbb{P}(|X_n| \leq K) = 1$ for some $K < \infty$ and all $n \geq 1$.

(a) Show that $|X| \leq K + 1$ with probability one.

(b) Show that $(X_n)_{n\geq 1}$ converges to X in L^1.

(c) Conclude that $\mathbb{E}[X_n] \to \mathbb{E}[X]$ as $n \to \infty$.

Problem 7. Let $n \geq 1$ be a positive integer, and consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω consists of all sequences $\omega = (\omega_1, \omega_1, \ldots, \omega_n)$ of zeros and ones of length n, $\mathcal{F}$ denotes the collection of all subsets of Ω, and $\mathbb{P}$ denotes the measure that assigns equal probability to the elements of Ω. For $k = 1, 2, \ldots, n$

(a) argue that $X_k(\omega) = \omega_k$ is a random variable;

(b) show that $\mathbb{P}(X_k = 1) = 1/2$;

(c) determine the σ-algebra generated by X_k.

Problem 8. Consider the following version of Polya's urn: Initially the urn contains one red, one blue and one green ball. At each round a ball is picked uniformly at random, and replaced together with another ball of the same colour. Let Y_n denote the proportion of red balls in the urn after n rounds.

(a) Show that $(Y_n)_{n\geq 0}$ defines a martingale with respect to itself.

(b) Show that $\lim_{n\to\infty} Y_n$ exists almost surely, and determine its mean.

(c) Suppose that the three balls initially present are marked in order to be distinguished from balls added later. Prove that each of the three balls initially present will be drawn infinitely many times each with probability one.

(d) Let T denote the number of rounds needed until the urn contains 10 red balls. Show that $\mathbb{E}[Y_T] = 1/3$.