STOCKHOLMS UNIVERSITET
Matematiska institutionen
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Re-exam / Omtentamen
MM7022 Logik II, 7,5 hp
HT 2021
Wed, 2022-02-02

This exam consists of 6 questions, worth a total of 40 points. Not all questions are equally difficult. You may submit your answers in either English or Swedish. Write clearly and motivate your answers carefully.

Good luck! — Lycka till!

1 Calculate the cardinalities of the following sets, and order them according to cardinality (some may be equal):

$$3^{\mathbb{N}}; \qquad \mathbb{N}^2; \qquad \mathbb{R}^{\mathbb{Q}}; \qquad \mathrm{Perm}(\mathbb{N}) \coloneqq \{f : \mathbb{N} \to \mathbb{N} \mid f \text{ is a bijection}\}.$$

2 Let L be a language, T a consistent theory over L.

(a) Show that if all models of T are elementarily equivalent, then T is (syntactically) complete.

(b) Show that if all models of T are isomorphic, then all models of T are finite.

3 (a) State the three main equivalent forms of the axiom choice: AC, Zorn's lemma and the well-ordering principle.

(b) Use one of them to prove the *Teichmüller-Tukey lemma*:

Let A be a set. Say a family F of subsets of A is of *finite character* if for every $U \subset A$, U belongs to F if and only if every finite subset of U belongs to F. Then the Teichmüller–Tukey lemma states: If $F \subseteq P(A)$ is of finite character, then for any $X \in F$, there is some maximal $Y \in F$ such that $X \subseteq Y$.

4 Recall that for a language L, a class of L-structures is *axiomatisable* (aka *definable*) if it is the class all models of some theory.

Let L be a language, and T a theory over L. For each $n \in \mathbb{N}$, write $\mathrm{Mod}_{<n}(T)$ for the class of all models of T of cardinality less than n, and $\mathrm{Mod}_{<\omega}(T) \coloneqq \bigcup_{n\in\mathbb{N}} \mathrm{Mod}_{<n}(T)$ for the class of all finite models of T.

(a) For each $n \in \mathbb{N}$, give a formula σ_n such that $T \cup \{F_n\}$ axiomatises $\mathrm{Mod}_{<n}(T)$.

(b) Show that if $\mathrm{Mod}_{<\omega}(T)$ is axiomatisable, then in fact $\mathrm{Mod}_{<\omega}(T) = \mathrm{Mod}_{<n}(T)$ for some $n \in \mathbb{N}$.

(c) Is $\mathrm{Mod}(T) \setminus \mathrm{Mod}_{<\omega}(T)$, the class of infinite models of T, axiomatisable?

5 Let L be the language consisting of a single binary relation $<$; consider the L-structure $\mathcal{N} \coloneqq \langle\, \mathbb{N} \,;\, < \,\rangle$.

(a) Show that every finite subset of $\mathcal{N}$ is L-definable. (That is: for each finite $X \subseteq \mathbb{N}$, there is some L-formula $\varphi_X(x)$ such that $[\![\, x \mid \varphi_X(x) \,]\!]^{\mathcal{N}} = X$.)

(b) Show that not every infinite subset of $\mathcal{N}$ is L-definable.

6 Let x be a set; let $\Gamma(x)$ be the class of all ordinals that admit some injection into x. Show the following properties:

(a) $\Gamma(x)$ is a set.

(b) $\Gamma(x)$ is an ordinal.

(c) $\Gamma(x)$ does not inject into x.

(d) $\Gamma(x)$ is the least ordinal that does not inject into x.

(You may assume earlier properties in later ones, if necessary.)

——— End of exam — Slut på provet ———