Solutions to Exam in Dynamical systems and optimal control theory, 2019-10-25

(1) We divide the proof into three parts. (i) "$\mathcal{R}_t \subseteq \mathcal{R}(A,B)$" : Pick up an arbitrary $\xi \in \mathcal{R}_t$. Now $\xi = x(t) = \int_0^t e^{A(t-s)}Bu(s)ds$. By the Caylay Hamilton Theorem, $e^{At} = \sum_{j=0}^{n-1} \alpha_j(t)A^j$. Hence

$$\xi = \int_0^t \sum_{j=0}^{n-1} \alpha_j(t-s)A^jBu(s)ds = (B, AB, ..., A^{n-1}B)\underbrace{\begin{pmatrix} \int_0^t \alpha_0(t-s)u(s)ds \\ \vdots \\ \int_0^t \alpha_{n-1}(t-s)u(s)ds \end{pmatrix}}_{\in \mathbb{R}^{n\times m}}$$

which means $\xi \in \mathcal{R}(A,B)$.

(ii) "$\mathcal{R}(A,B) \subseteq \mathrm{Im}W(0,t)$" where $W(0,t) = \int_0^t e^{A(t-s)}BB'e^{A'(t-s)}ds$: Pick up $\xi_1 \in \mathcal{R}$. Then there exists $\eta \in \mathbb{R}^{nm}$ such that $R(A,B)\eta = \xi_1$. Assume $\xi_1 \notin \mathrm{Im}W(o,t)$ for same $t > 0$. We shall show that this leads to a contradiction. Indeed this assumption implies that the $\ker W(0,t)$ is not empty and therefore there exits a nonzero $\xi_2 \in \mathbb{R}^n$ such that $W(0,t)\xi_2 = 0$. Note that $\xi_2'\xi_1 \neq 0$ (for otherwise $\xi_1 \in \mathrm{Im}W(0,t)' = \mathrm{Im}W(0,t)$ since $W(0,t)$ is symmetric, which is not true by assumption). Now consider

$$0 = \xi_2'W(0,t)\xi_2 = \int_0^t \xi_2'e^{(t-s)A}BB'e^{(t-s)A'}\xi_2 ds = \int_0^t \|\xi_2'e^{(t-s)A}B\|^2 ds.$$

implying $\xi_2'e^{(t-s)A}B = 0$ for all $s \in [0,t]$. Successive differentiation of both sides w.r.t. s and evaluate at $s = t$ gives

$$\xi_2'B = -\xi_2'AB = ... = (-1)^k\xi_2'A^kB = 0 \quad \forall k > 0, \Leftrightarrow \xi_2'A^kB = 0\ \forall k \geq 0$$
$$\Rightarrow \xi_2'\xi_1 = \xi_2'R(A,B)\eta = 0,$$

a contradiction, which shows that $\xi_1 \in \mathrm{Im}W(0,t)$.

(iii) "$\mathrm{Im}W(0,t) \subseteq \mathcal{R}_t$": Take any $\xi \in \mathrm{Im}W(0,t)$. Then there exits an $\eta \in \mathbb{R}^n$ such that $W(0,t)\eta = \xi$. Define $u = B'e^{A'(t-s)}\eta$ for all $0 \leq s \leq t$. Then the solution to $\dot{x} = Ax + Bu$ with $x(0) = 0$ at t is

$$x(t) = \int_0^t e^{A(t-s)}Bu(s)ds = \int_0^t e^{A(t-s)}BB'e^{A'(t-s)}\eta ds = W(0,t)\eta = \xi$$

which implies, by definition, $\xi \in \mathcal{R}_t$. Now we have $\mathcal{R}_t \subseteq \mathcal{R}(A,B) \subseteq \mathrm{Im}W(0,t) \subseteq \mathcal{R}_t$. This completes the proof.

(2) Let $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$. Then $R(A,b) = (b, Ab, A^2b, A^3b) = \begin{pmatrix} 0 & 1 & 4 & 12 \\ 1 & 2 & 4 & 8 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ has rank 3. And the system is already in the decomposed form. The subsystem of the first three equations is controllable and the uncontrollable mode is -1. So the system is asymptotically controllable and thus stabilizable. A state feedback can be $u = Kx$ with $K = (-16, -8, 0, 0)$. Since the system is not completely controllable so it is not possible to place all eigenvalues at 2.

(3) Pick up $z \in \ker(K) \Leftrightarrow Kz = 0$. Then

$$0 = -A'Kz - KAz + KLKz - C'Cz = -KAz - C'Cz \qquad (*)$$

and thus

$$0 = -z'KAz - z'C'Cz = -z'C'Cz \;\Leftrightarrow\; Cz = 0$$

Now (*) becomes $KAz = 0$, i.e. $Az \in \ker(K)$, that is $\ker(K)$ is A-invariant. Multiplying the (ARE) by Az from left gives

$$0 = -A'KAz - KA^2z + KLKAz - C'CAz = -KAz - C'Cz = -KA^2 - C'CAz$$

$$\Rightarrow 0 = \underbrace{-z'A'K}_{0} A^2z - z'A'C'CAz, \ \Rightarrow \ CAz = 0 \ \Rightarrow \ KA^2z = 0 \ \Rightarrow \ A^2z \in \ker(K)$$

Continue in the similar manner we obtain $A^kz \in \ker(K)$, for $k = 1, 2, ..., n-1$. This yields

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} z = 0 \Rightarrow \ z \in \ker \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

Since (A, C) is observable, z must be zero so the $\ker(K)$ is trivial, i.e., K is nonsingular.

(4) Solving the equations $-x_1 + g(x_2) = 0$, $\dot{x}_2 = -x_2 + h(x_1) = 0$ we obtain a unique equilibrium at the origin. Consider next the function $V(x_1, x_2) = x_1^2/2 + x_2^2/2$. Obviously V and its partial derivatives are continuous and $V(x_{1,2}) > 00$ for all $0 \neq (x_1, x_2) \in \mathbb{R}^2$. Now we compute $\dot{V}$.

$$\dot{V}(x_1, x_2) = -x_1^2 - x_2^2 + x_1g(x_2) + x_2h(x_1) \leq -x_1^2 - x_2^2 + |x_1x_2|/2 + |x_1x_2|/2$$

$$= -x_1^2 - x_2^2 + |x_1x_2| \leq -x_1^2 - x_2^2 + \frac{1}{2}(x_1^2 + x_2^2) - \frac{1}{2}(x_1^2 + x_2^2) = -V(x_1, x_2) < 0$$

for all $0 \neq (x_1, x_2) \in \mathbb{R}^2$. So the systemis globally asymptotically stable.

(5) Here is a "raw" computation. Note that we only have to compute the $(1,1)$-element in $(sI - A)^{-1}$ because only the first element in b and c is non-zero. I use the Cramér rule and multiply then $1/g_1$ to get $c'(sI - A)^{-1}b = \frac{p(s)}{q(s)}$ where

$$p(s) = s^2 + (\frac{1}{g_2g_3} + \frac{1}{g_3g_4} + \frac{1}{g_4g_5} + \frac{1}{g_5g_6})s + \frac{1}{g_2g_3g_4g_5} + \frac{1}{g_2g_3g_5g_6} + \frac{1}{g_3g_4g_5g_6}$$

$$q(s) = g_1s^3 + (\frac{1}{g_2} + \frac{g_1}{g_2g_3} + \frac{g_1}{g_3g_4} + \frac{g_1}{g_4g_5} + \frac{g_1}{g_5g_6})s^2 + (\frac{1}{g_2g_3g_4} + \frac{1}{g_2g_4g_5} + \frac{g_1}{g_2g_3g_4g_5} + \frac{1}{g_2g_5g_6} + \frac{g_1}{g_2g_3g_5g_6} + \frac{g_1}{g_3g_4g_5g_6})s + \frac{1}{g_2g_3g_4g_5g_6}$$

Then

$$\frac{p(s)}{q(s)} = \frac{1}{q(s)/p(s)} = \frac{1}{g_1s + \frac{p_2(s)}{p(s)}} = \frac{1}{g_1s + \frac{1}{p(s)/p_2(s)}}$$

where $p_2(s) = \frac{s^2}{g_2} + (\frac{1}{g_2g_3g_4} + \frac{1}{g_2g_4g_5} + \frac{1}{g_2g_5g_6})s + \frac{1}{g_2g_3g_4g_5g_6}$

$$\frac{p(s)}{p_2(s)} = \frac{1}{g_2 + \frac{p_3(s)}{p_2(s)}} = \frac{1}{g_2 + \frac{1}{p_2(s)/p_3(s)}}, \text{ where } p_3(s) = \frac{s}{g_2g_3} + \frac{1}{g_2g_3g_4g_5} + \frac{1}{g_2g_3g_5g_6}$$

$$\frac{p_2(s)}{p_3(s)} = g_3s + \frac{\frac{\frac{s}{g_2g_3} + \frac{1}{g_2g_3g_5g_6} + \frac{1}{g_2g_3g_4g_5}}{g_4} - \frac{1}{g_2g_3g_4^2g_5}}{p_3(s)} =: g_3s + \frac{p_4(s)}{p_3(s)} = g_3s + \frac{1}{p_3(s)/p_4(s)}$$

$$\frac{p_3(s)}{p_4(s)} = g_4 + \frac{\frac{1}{g_2g_3g_4g_5}}{p_4(s)} = g_4 + \frac{1}{g_5s + \frac{1}{g_6}}$$

It's easy to check that $(Ab, c) \in \mathcal{S}_{n,1,1}^{\text{contra.Obs}}$ so the realization is minimal. And the given sequence can be realized by the above realization with $g_i = 1$, $i = 1, ..., 6$