(1) Solving the equations $-x_1 + g(x_2) = 0$, $\dot{x}_2 = -x_2 + h(x_1) = 0$ we obtain a unique equilibrium at the origin. Consider next the function $V(x_1, x_2) = x_1^2/2 + x_2^2/2$. Obviously V and its partial derivatives are continuous and $V(x_{1,2}) > 00$ for all $0 \neq (x_1, x_2) \in \mathbb{R}^2$. Now we compute $\dot{V}$.

$$\dot{V}(x_1, x_2) = -x_1^2 - x_2^2 + x_1 g(x_2) + x_2 h(x_1) \leq -x_1^2 - x_2^2 + |x_1 x_2|/2 + |x_1 x_2|/2$$
$$= -x_1^2 - x_2^2 + |x_1 x_2| \leq -x_1^2 - x_2^2 + \frac{1}{2}(x_1^2 + x_2^2) - \frac{1}{2}(x_1^2 + x_2^2) = -V(x_1, x_2) < 0$$

for all $0 \neq (x_1, x_2) \in \mathbb{R}^2$. So the systemis globally asymptotically stable.

(2) (a) is trivial. Linearizing the system at $(0, w_2^*, 0)$ we get

$$I_1 \frac{d}{dt}\Delta w_1 = (I_2 - I_3) w_2^* \Delta w_3 + N_1$$
$$I_2 \frac{d}{dt}\Delta w_2 = N_2$$
$$I_3 \frac{d}{dt}\Delta w_3 = (I_1 - I_2) w_2^* \Delta w_1 + N_3$$

It is not stable if we don't use the torque because the eigenvalues

$$0, \pm w_2^* \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}}$$

are not all in the left half plane. So the open loop system is unstable.

Since this is a controllable system (the input matrix i diagonal matrix with diagonal elements $1/I_1, 1/I_2, 1/I_3$ which has full rank) it is possible to stabilize the system. However it is a multi-input system. A straightforward calculation show that it is not enough to just use one torque and we need at least two and a quick check show that two tongues are enough, for example N_1, N_2, that is set $N_3 = 0$. Let $u_1 = N_1/I_1$, $u_2 = N_2/I_2$. We should choose a feedback $K = (k_{ij})$ (a 2×3 matrix) such that the close loop system has all poles at $w_2^* \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}}$. Since the second row of A matrix is zero so we can choose $k_{21} = k_{23} = 0$ and k_{22} the pole we want to put at. Thus we get

$$N_2 = I_2 u_2 = k_{22} \Delta w_2 = -w_2^* I_2 \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}} \Delta w_2$$

Consequently we can choose $k_{12} = 0$ to work with a low dimensional system, the pair $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with $a = (I_2 - I_3) w_2^*/I_1$, $b = (I_1 - I_2) w_2^*/I_3$. We want to determine the feedback gain matrix (k_{11}, k_{13}). It is not hard to find, by inspection, that $k_{11} = -2w_2^* \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}}$ and $k_{13} = -2w_2^*(I_3 - I_2)/I_1$. Thus

$$N_1 = -2 I_1 w_2^* \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}} \Delta w_1 - 2 w_2^* (I_3 - I_2) \Delta w_3$$

So the feedback law

$$N_1 = -2 I_1 w_2^* \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}} \Delta w_1 - 2 w_2^* (I_3 - I_2) \Delta w_3, \ N_2 = 0, \ N_3 = 0$$

will put all the closed loop poles at $-w_2^*\sqrt{\frac{(I_3-I_2)(I_2-I_1)}{I_1I_3}}$.

(3) (a) Since C is invertible $A+BNC = C^{-1}(CAC^{-1}+CBN)C$. The eigenvalues of $A+NBC$ are the same as those of $CAC^{-1}+(CB)N$. Since (CAC^{-1}, CB) is equivalent to (A,B) which is controllable, (CAC^{-1}, CB) is controllable. So there is an N such that the matrix $CAC^{-1}+(CB)N$ is Hurwitz.
(b) can be done in the same way.
(c) Let

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, c = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$$

Clearly (A,b) is controllable and (A,c) is observable. By the matrix dimension N must be a scalar. So $A+BNC$ is the same as A except the element in the position $(n,1)$ which is N. So the characteristic polynomial of $A+BNC$ is $s^n - N$. So $A+BNC$ will never be Hurwitz for otherwise all coefficients of the characteristic polynomial must be positive.
(d) Note that in the previous example $cb = 0$ and A has trace 0 which imply that the trace of $A+BNC$ is zero. Moreover the trace of the matrix is negative of the coefficient of s^{n-1} So in general if $p+m \le n$ if $CB = 0$ and the trac of A is zero then we can prove that the matrix $A+BNC$ will never be Hurwitz. This follows by the fact that the $\mathrm{tr}(A+BNC) = \mathrm{tr}(A+CBN) = \mathrm{tr}(A) = 0$.

(4) It is apparent the system is not stable without control since not all eigenvalues $2, -1$ are on the left half plane. Note that the system can be divided into two subsystems where $\dot{x}_4 = -z$ is not controllable. But the subsystem for x_1, x_2, x_3 is. Clearly it is impossible to place poles at $-2,-2,-2,-2$ because nothing can influence the trajectory of x_4. Since the subsystem of x_1, x_2, x_3 is controllable and the uncontrollable mode is -1 on the left half plane so it is possible to place poles at $-2,-2,-1,-1$ and $u = \begin{pmatrix} -18 & -8 & 0 & 0 \end{pmatrix} x + v$ is a feedback.

(5) To simplify notation we drop t in the computations below. Note that

$$\frac{d}{dt}K^{-1} = -K^{-1}\frac{dK}{dt}K^{-1}.$$

Plug in the Riccati equation we get

$$\frac{d}{dt}K^{-1} = K^{-1}A' + AK^{-1} - BB'$$

and $K^{-1}(t_1) = Q^{-1}$. Using the variation of constants formula for the matrix equations we can write down the solution for K^{-1}:

$$\begin{aligned} K^{-1}(t) &= \Phi(t,t_1)K^{-1}(t_1)\Phi(t,t_1)' - \int_{t_1}^{t} \Phi(t,s)BB'\Phi(t,s)'ds \\ &= \Phi(t,t_1)K^{-1}(t_1)\Phi(t,t_1)' + \int_{t}^{t_1} \Phi(t,s)BB'\Phi(t,s)'ds \\ &= \Phi(t,t_1)Q^{-1}\Phi(t,t_1)' + W(t,t_1) \end{aligned}$$

The optimization problem in question is an LQ control problem so the optimal solution for u is

$$u(t) = -B'K(t)x(t) = -B'(\Phi(t,t_1)Q^{-1}\Phi(t,t_1)' + W(t,t_1))^{-1}x(t)$$

provided the inverse exists.

(6) (a) We check the rank of $(sI - A_1, b)$.

$$(sI - A_1, b) = \begin{pmatrix} s-\lambda & 0 & 0 & b_1 \\ 0 & s-\lambda & -1 & b_2 \\ 0 & 0 & s-\lambda & b_3 \end{pmatrix}$$

Take $s = \lambda$ we see that the rank is not 3 so the pair is always uncontrollable.

(b) We would like to find condition on c so that $\begin{pmatrix} sI - A_2 \\ c \end{pmatrix}$ has full rank.

$$\begin{pmatrix} sI - A_2 \\ c \end{pmatrix} = \begin{pmatrix} s-\lambda & -1 & 0 \\ 0 & s-\lambda & -1 \\ 0 & 0 & s-\lambda \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

When $s = \lambda$ we see that the rank will be 3 if and only if $c_1 \neq 0$.

(c) whenever more than one independent eigenvectors can be associated with a single eigenvalue we shall loose controllability (and observability). This follows from the Hautus lemma applied to the equivalent system in the Jordan canonical form.

(d) $(A, b, c) \in \mathcal{S}_{n,1,1}$ is minimal is equivalent to (A, b) is controllable and (A, c) is observable. If $a(s) = \det(sI - A)$ has a repeated root, that is, A has multiple eigenvalues then A could be diagonalized by a similarity transformation means it has an equivalent minimal system with the diagonal system matrix. Then there would be more than one independent eigenvectors associated with the repeated eigenvalue. By (c) we shall loose controllability (and observability).