MATEMATISKA INSTITUTIONEN
STOCKHOLM UNIVERSITET
Avd. Matematik
Examinator: Yishao Zhou

Tentamensskrivning i
Dynamiska system och
optimal kontrollteori, AN
den 29 oktober 2018

Open book exam: Sontag, Mathematical Control Theory, Deterministic finite dimensional systems, 2nd edition, Springer. No other material is allowed.
There are 6 problems with 10 points each. They are not arranged in degree of difficulty.
The mark E requires at least 12 points together with the successful homework and project work. Tentatively, D, C, B and A require at least 24p, 36p, 46p and 56p, respectively.

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(1) Consider the nonlinear system

$$\dot{x}_1 = -x_1 + g(x_2)$$
$$\dot{x}_2 = -x_2 + h(x_1)$$

where $g(0) = h(0) = 0$, and $|g(u)| \leq |u|/2$ and $|h(u)| \leq |u|/2$. Show that it is globally asymptotically stable.

(2) Consider the dynamical system describing motion of a spinning body

$$I_1\dot{w}_1 = (I_2 - I_3)w_2w_3 + N_1$$
$$I_2\dot{w}_2 = (I_3 - I_1)w_3w_1 + N_2$$
$$I_3\dot{w}_3 = (I_1 - I_2)w_1w_2 + N_3$$

with $0 < I_1 < I_2 < I_3$. Here w_1, w_2, w_3 denote the rate of spinning of the body around its principal axes, and the acceleration torques N_1, N_2, N_3 are inputs which can be achieved for example by electromotors mounted on the main axis of the spinning body.

(a) Show that $(w_1^*, 0, 0)$, $(0, w_2^*, 0)$, $(0, 0, w_3^*)$ are equilibrium points when $N_1 = N_2 = N_3 = 0$, where $w_i^* > 0$, $i = 1, 2, 3.$.
(b) Show that the this body cannot spin in an asymptotically stable mode in the absence of the inputs.
(c) Examine if this asymptotical stability can be achieved by exerting control using the linear theory (you only need to do so around one equilibrium point from (a)).
(d) In case possible determine a state feedback that puts all poles at the negative eigenvalues of the linearized system matrix.

(3) Consider the time-invariant closed-loop system

$$\dot{x} = (A + BNC)x, y = Cx, u = NCx$$

where $(A, B, C) \in \mathcal{S}_{n,m,p}$. The question arises whether for a given (A, B, C) there exists an N such that $A + BNC$ is Hurwitz. We call this *stabilizability by memoryless output feedback*.

(a) Prove that if (A, B) is controllable, $p = n$ and $\det C \neq 0$, then stabilizability by memoryless output feedback is possible.
(b) Prove that if (A, C) is observable $m = n$ and $\det B \neq 0$, then stabilizability by memoryless output feedback is possible.
(c) Prove that if $m = p = 1$ and $n > 1$ there exists $(A, B, C) \in \mathcal{S}_{n,1,1}^{\text{cont,obs}}$ and B, C both of full rank which cannot be stabilized by memoryless output feedback. (You can skip this part if you can solve the next problem correctly.)
(d) Generalize the previous statement to $m + p \leq n$.

(4) Consider

$$\frac{dx}{dt} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} u$$

Is the system without control stable? (motivate your answer). Is it possible to find a vector K such that the system with the feedback $u = Kx + v$ has its poles at $-2, -2, -1, -1$. Determine K if the answer is affirmative otherwise show that it is impossible. Is it possible to place all poles at -2?

(5) Let A, B and $Q = Q'$ be given constant matrices of appropriate dimensions. Assume that $\Phi(t, s)$ is the state transition matrix of $\dot{x} = Ax$ and $W(t_0, t_1)$ is the controllability granmmian for $\dot{x} = Ax + Bu$, $t_0 \leq t \leq t_1$. Consider the Riccati differential equation for $t_0 \leq t \leq t_1$,

$$\dot{K}(t) = -A'K(t) - KA + KBB'K, K(t_1) = Q.$$

(a) Show that if $K(t)$ is invertible then $K^{-1}(t)$ satisfies a differential Lyapunov equation.
(b) Show that $K^{-1}(t) = \Phi(t, t_1)Q\Phi(t, t_1)' + W(t, t_1)$
(c) Let $\eta = \int_{t_0}^{t_1} u(t)'u(t)dt + x(t_1)'Qx(t_1)$. Find u (in terms of the results from previous partial problems) such that η is minimized subject to $\dot{x}(t) = Ax(t) + Bu(t)$, $x(t_0) = x_0$ given

(6) (a) Show that any pair $(A_1, b) \in \mathcal{S}_{3,1}$ with

$$A_1 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

is always uncontrollable.
(b) Find condition on c so that (A_2, c) will always be observable when

$$A_2 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

(c) What conclusion will you draw in context of controllability and observability if more than one independent eigenvector can be associated with a single eigenvalue of A? Justify your conclusion.
(d) Assume $(A, b, c) \in \mathcal{S}_{n,1,1}$ is minimal and that $a(s) = \det(sI - A)$ has a repeated root. Prove that A cannot be diagonalized by a similarity transformation.

Skrivningsåterlämning äger rum i rum 403, hus den 8 november 2018 kl 11.00, därefter på studentexpedition.