MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sofia Tirabassi

Exam in
Combinatorics
7.5 hp
January 5th, 2023

Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- A maximum score of 30 points can be achieved. A score of at least 15 points will ensure a pass grade.

GOOD LUCK!

1. **Counting:** Let a_n the number of ways you can give n Euros in banknotes of 10, 20, 50, 100 and 500 Euros.

   (a) (2 points) Find the generating function for a_n, and express it as a rational function.

   (b) (3 points) Compute a_{100}.

2. **Rook polynomials:** Remember that the rook polynomial of a $m \times n$ chessboard without forbidden cells is
$$r(C,x) = \sum_{k=0}^{m} \binom{n}{k}\binom{m}{k} k! x^k$$
   Fix $n \geq m$ two positive integers. For $i \leq m$ and let C_i be an $n \times m$ in which the first column has exactly i allowed places.

   (a) (2 points) Compute the rook polynomial of C_2 when $m = 2$ and $n = 3$.

   (b) (3points) Find a formula for the rook polynomial for C_3 for any n and m.

3. **Recursion:** (5 points) Consider the following recursion relation
$$a_{n+2} + a_{n+1} + a_n = n$$
   with boundary conditions $a_0 = 0$ and $a_1 = 1$. Solve the relation finding a closed formula for a_n.

4. **Graphs:** We want to use graph theory to solve the following problem: The mathematics department has 6 committees that meets once a month:

   - $C_1 = \{A, B, Z\}$
   - $C_2 = \{B, L, R\}$
   - $C_3 = \{A, R, Z\}$
   - $C_4 = \{L, R, Z\}$
   - $C_5 = \{A, B\}$
   - $C_6 = \{B, R, Z\}$

   How many different meeting times have to be scheduled to ensure that no member is scheduled to attend two meetings at the same time?
   Let $G = (V, E)$ the graphs whose vertices are the committees and two committees are adjacent if, and only if, the intersection of their members is not empty.

   (a) (1 point) Make a sketch of G and determine the degree of each vertex.

   (b) (1 point) Decide if G has an Euler path.

   (c) (2 point) Decide if G is planar.

   (d) (1 points) Compute the chromatic number of G (this answer our leading question).

   (e) (2 points) Compute the chromatic polynomial of G.

5. **Minimal spanning trees:** (3 points) With the graph froms the previous exercise, consider the weight function $W : E(G) \to \mathbb{R}_{>0}$ defined by $W(\{C_i, C_j\}) = |C_i \cap C_j|$.
   Find a minimal spanning tree and give its total weight. In order to get full points you have to state which algorithm you are using (Kruskal or Prym) and show the iterations at every step.

6. **Transport Networks:** Consider the transport network in Figure 1, where S is the source and T is the sink.

   (a) (3 points) Find a flow with the maximum value for the network.

   (b) (2 points) Give a cut with the minimum capacity for the network. Determine the capacity of such cut.

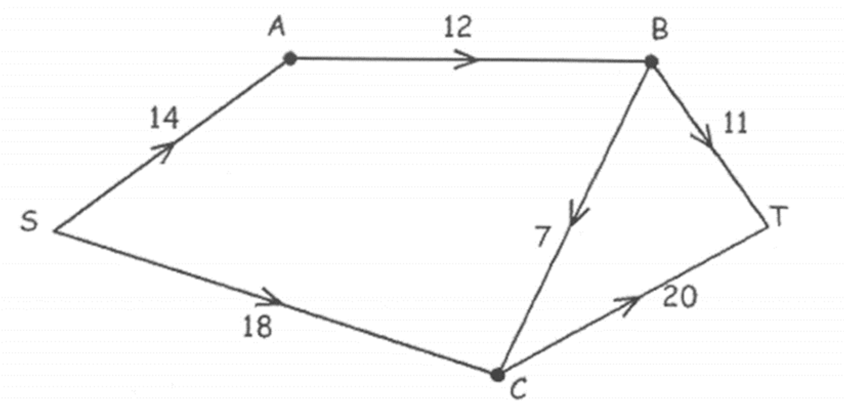

Figure 1: Network