STOCKHOLMS UNIVERSITET,
MATEMATISKA INSTITUTIONEN,
Avd. Matematisk statistik

Exam: Brownian motion and stochastic differential equations (MT7043), 2023-01-10

Examiner:
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Allowed aid: Calculator (provided by the department).

Return of exam: To be announced via the course webpage or the course forum.

The exam consists of five problems. Each problem gives a maximum of 10 points.

- The reasoning should be clear and concise.
- Answers should be motivated (unless otherwise stated).
- Any assumptions should be clearly stated and motivated.
- Start every problem on a new sheet of paper.
- Clearly number each sheet with problem number and sheet order.
- Write your code number (but no name) on each sheet.

Preliminary grading:

A	B	C	D	E
46	41	36	30	25

Good luck!

Problem 1

(A) Let Ω be a given set. State the definition of a σ-algebra $\mathcal{F}$ on Ω. (3p)

(B) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For this probability space, state the definition of a random variable Y and a one-dimensional continuous time stochastic process $(X_t)_{t\geq 0}$. (4p)

(C) State the martingale representation theorem. (Note that you are not asked to give a proof). (3p)

Problem 2

(A) Find the SDE that the process given by

$$e^{-\frac{1}{2}t+B_t}$$

solves. (Do not forget that answers should be motivated).

Hint/suggestion: use Itô's formula. (5p)

(B) Consider a one-dimensional Brownian motion and a function $f : \mathbb{R} \to \mathbb{R}$ with $f \in C_0^2$. Which equation (including boundary condition(s)) does the function u defined by

$$u(t, x) := \mathbb{E}^x(f(B_t))$$

satisfy? (No motivation is needed for this part of the question; but the equation should be explicitly stated.) (5p)

Problem 3

(A) Is the process given by

$$B_t^3 - 3tB_t$$

a martingale?

Hint/suggestion: use Itô's formula. (4p)

(B) Is the process (Y_t) given by

$$Y_t := \frac{B_{4t}}{4} \tag{1}$$

a martingale? (3p)

(C) Is the process (Y_t) given by (1) a Brownian motion? (3p)

Problem 4

Consider an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ with $f \in C_0^2$. Let (B_t) be a one-dimensional Brownian motion and $s \in (0, \infty)$ be a constant.

Is it possible to find a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}^x \left(f(B_s) \right) = f(x) + \mathbb{E}^x \left(\int_0^s g(B_u) du \right) \tag{2}$$

?

If your answer is yes, then give an expression for the function g and prove that (2) holds for this specific choice of g.

If your answer is no, then show that your answer is correct.

(10p)

Hint/suggestion: use Itô's formula and properties of Itô integrals.

Problem 5

Let (B_t) be a one-dimensional Brownian motion. Find a candidate solution to the optimal stopping problem

$$V(x) = \sup_{\tau} J^{\tau}(x)$$

where

$$J^{\tau}(x) = \mathbb{E}^x \left(e^{-r\tau} |B_{\tau}| \right), \quad \text{with } r = \frac{1}{2}.$$

Hints/suggestions/instructions:

1. Use the ansatz that the optimal solution is found by stopping outside of the interval $(-x_0, x_0)$, for some constant $x_0 > 0$.
2. **A complete solution involves**

   (i) finding/characterizing an appropriate candidate for the optimal value x_0, and

   (ii) determining (as explicitly as possible) the corresponding optimal value function.
3. Recall that the solution to the ODE

   $$\frac{1}{2} f''(x) - r f(x) = 0$$

   when $r = \frac{1}{2}$ is

   $$Ae^x + Be^{-x}.$$

(10p)