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Exam: Brownian motion and stochastic differential equations (MT7043), 2023-01-10

Problem 1

(A) See Øksendal ch. 2.1.

(B) See Øksendal ch. 2.1.

(C) See Øksendal ch. 4.3.

Problem 2

(A) Using Itô's formula we find with some calculations that $f(t, B_t) := e^{-\frac{1}{2}t+B_t}$ satisfies

$$\begin{aligned} df(t,B_t) &= f_t(t,B_t)dt + \frac{1}{2}f_{xx}(t,B_t)dt + f_x(t,B_t)dB_t \\ &= -\frac{1}{2}f(t,B_t)dt + \frac{1}{2}f(t,B_t)dt + f(t,B_t)dB_t \\ &= f(t,B_t)dB_t. \end{aligned}$$

Hence, it is clear that $e^{-\frac{1}{2}t+B_t}$ solves the SDE

$$dX_t = X_t dB_t, \quad X_0 = 1.$$

(B) Using Øksendal Theorem 8.1.1(a) we find that

$$\begin{cases} u_t(t,x) & = \frac{1}{2}u_{xx}(t,x), \quad t > 0, x \in \mathbb{R} \\ u(0,x) & = f(x), x \in \mathbb{R}. \end{cases}$$

The above relies on $A_X u = L_X u$, which can be established using that B_t is normally distributed (the question could also have been answered without this observation).

Problem 3

(A) Using Itô's formula we find with some calculations that $f(t, B_t) := B_t^3 - 3tB_t$ satisfies

$$\begin{aligned} df(t,B_t) &= f_t(t,B_t)dt + \frac{1}{2}f_{xx}(t,B_t)dt + f_x(t,B_t)dB_t \\ &= -3B_t dt + 6B_t\frac{1}{2}dt + (3B_t^2 - 3t)dB_t \\ &= (3B_t^2 - 3t)dB_t. \end{aligned}$$

Hence, our process can be written as an Itô integral and is a martingale (with respect to the filtration generated by (B_t)).

(B) Since $X_t := 2Y_t = \frac{B_{4t}}{2}, t \geq 0$ is a Brownian motion (cf. Øksendal p. 19 or Exercise document 2) it holds that (X_t) is a martingale (with respect to the filtration generated by (X_t)). From this it is directly seen that also (Y_t) is a martingale (with respect to the same filtration).

(C) It is not a Brownian motion, which can be seen by noting that $V(\frac{B_{4t}}{4}) = \frac{1}{16}V(B_{4t}) = \frac{1}{16}4t \neq t$.

Problem 4

Using Itô's formula we find

$$\begin{aligned}\mathbb{E}^x\left(f(B_s)\right) &= \mathbb{E}^x\left(f(x) + \int_0^s f'(B_u)dB_u + \int_0^s \frac{1}{2}f''(B_u)du\right)\\ &= f(x) + \mathbb{E}^x\left(\int_0^s \frac{1}{2}f''(B_u)du\right).\end{aligned}$$

Hence, the statement in the problem is true when we define g according to $g(x) = \frac{1}{2}f''(x)$.

Note that the Itô integral in the above vanished since it is a martingale; cf. Corollary 3.2.6 (the condition that the process $f''(B_u), u \geq 0$ is a member of $\mathcal{V}$ can be directly verified using e.g., that $f \in C_0^2$.)

Problem 5

(Note that the problem can be solved in different ways). Let $D = (-x_0, x_0)$ and $\tau_D = \inf\{t : X_t \notin D\}$. In line the examples that we have seen during the course our ansatz boils down to identifying the (candidate) corresponding value function as

$$J^{\tau_D}(x) = \left\{\begin{array}{ll} f(x), & x \in (-x_0, x_0)\\ |x|, & x \in (-x_0, x_0)\end{array}\right.$$

where f is chosen so that $-rf(x) + Lf(x) = 0$ (where L is the differential operator corresponding to (B_t), i.e., $L = \frac{1}{2}\frac{d^2}{dx^2}$) and $x \mapsto J^{\tau_D}(x)$ is continuous. This means that $f(x)$ is given by the hint in the question with A and B chosen so that

$$J^{\tau_D}(x) = \begin{cases} x_0\frac{e^x+e^{-x}}{e^{x_0}+e^{-x_0}}, & x \in (-x_0, x_0)\\ |x|, & x \notin (-x_0, x_0).\end{cases} \tag{1}$$

(To see the above use the hint, continuity and basic calculations).

Moreover, the optimal value for x_0 is identified using smooth fit; which in the present case (since $\frac{d}{dx}|x| = 1$ for $x > 0$ and $\frac{d}{dx}|x| = -1$ for $x < 0$) means finding x_0 so that the two conditions

$$J^{\tau_D\prime}(x_0) = 1,$$

$$J^{\tau_D}{}'(-x_0) = -1$$

are satisfied. Using basic calculations we see that these two conditions are attained by the same value for x_0 (which is not surprising since our problem is in a sense symmetric around 0) and in particular we find that smooth fit is attained when choosing x_0 so that

$$x_0 \frac{e^{x_0} - e^{-x_0}}{e^{x_0} + e^{-x_0}} = 1. \tag{2}$$

Our candidate for the optimal solution is hence given by setting x_0 to be the positive solution to the equation (2) (it is possible to show that this equation has a unique positive solution which is $x_0 \approx 1.19968$, but this is not required in your solution).

We have thus found a candidate solution for the optimal stopping time to be

$$\tau_D = \inf\{t : X_t \notin D\}$$

with $D = (-x_0, x_0)$, and x_0 defined as the positive solution to (2). The corresponding optimal value function is hence (1) with x_0 set to the positive solution to (2). (It can be verified that this is indeed an optimal solution.)