MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Tentamensskrivning i
Topologi, MM7052
7.5 hp
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(1) Let I be an operation on the set of all subsets of a set X such that
   (i) $I(X) = X$,
   (ii) $I(A) \subseteq A$,
   (iii) $I(I(A)) = I(A)$,
   (iv) $I(A \cap B) = I(A) \cap I(B)$,

   for all $A, B \subseteq X$. Declare $A \subseteq X$ to be open if $I(A) = A$.
   (a) Show that this defines a topology on X. (3p)
   (b) Show that every topology on X is induced by an operation I as above. (2p)

(2) Consider the graph of a function $f\colon X \to Y$ between topological spaces,
$$\Gamma(f) = \{(x, y) \in X \times Y \mid y = f(x)\}.$$
   (a) Suppose that Y is Hausdorff. Show that if f is continuous, then $\Gamma(f)$ is closed in $X \times Y$. (3p)
   (b) Suppose that Y is compact. Show that if $\Gamma(f)$ is closed in $X \times Y$, then f is continuous. (3p)

   (Hint: For (b) you may use the fact that the projection $X \times Y \to X$ is a closed map if Y is compact.)

(3) Let A and B be subsets of a topological space X.
   (a) Show that if A and B are closed, and $A \cup B$ and $A \cap B$ are connected, then A and B are connected. (4p)
   (b) Give an example to show that this is not necessarily true if one of the sets A, B is not closed. (2p)

(4) Consider the configuration space of two points on the sphere,
$$\mathfrak{C}_2(S^2) = \big\{(x, y) \in S^2 \times S^2 \mid x \neq y\big\}.$$
   (a) Show that $\mathfrak{C}_2(S^2)$ is homotopy equivalent to S^2. (3p)
   (b) The unordered configuration space is the orbit space
$$\mathfrak{C}_2^u(S^2) = \mathfrak{C}_2(S^2)/C_2$$
   where the cyclic group C_2 of order 2 acts by permuting the two points. Compute the fundamental group of $\mathfrak{C}_2^u(S^2)$. (3p)

(5) Let n be a positive integer and let X_n be the quotient space obtained from the unit disk in the complex plane,
$$D^2 = \big\{z \in \mathbb{C} \mid |z| \leq 1\big\},$$
   by identifying points on the boundary that are $2\pi/n$ radians apart, i.e., we identify z with $e^{2\pi i/n}z$ if $|z| = 1$.
   (a) Compute the fundamental group of X_n. (3p)
   (b) For which n is every non-trivial covering space of X_n simply connected? (2p)
   (c) For which n is X_n a surface? (2p)