Topology MM7052, HT22.

Solutions to Exam 2023-01-24

(1) (a) X is open by (i) and $\emptyset$ is open by (ii) since $I(\emptyset) \subseteq \emptyset$ implies $I(\emptyset) = \emptyset$. If A and B are open, then by (iv) we have $I(A \cap B) = I(A) \cap I(B) = A \cap B$, showing $A \cap B$ is open. To prove that arbitrary unions of open sets are open, first note that $A \subseteq B$ implies $I(A) \subseteq I(B)$, because $A \subseteq B \Leftrightarrow A = A \cap B \Rightarrow I(A) = I(A \cap B) = I(A) \cap I(B) \Leftrightarrow I(A) \subseteq I(B)$. Now let $\{A_i\}_i$ is a family of open sets. By (ii) we have $I(\cup_i A_i) \subseteq \cup_i A_i$. To show the reverse inclusion, note that $A_i \subseteq \cup_i A_i$ implies $A_i = I(A_i) \subseteq I(\cup_i A_i)$ by the above, whence $\cup_i A_i \subseteq I(\cup_i A_i)$.

(b) Given a topology on X, define $I(A)$ to be the interior of A, i.e., the union of all open sets contained in A. Then we clearly have that $I(A)$ is open and that $I(A) \subseteq A$ with equality if and only if A is open. (i), (ii), and (iii) follow immediately from this. To show (iv), note that $A \subseteq B$ implies $I(A) \subseteq I(B)$. This in turn implies $I(A \cap B) \subseteq I(A) \cap I(B)$. The reverse inclusion follows from the fact that $I(A) \cap I(B)$ is open and contained in $A \cap B$.

(2) (a) First note that Y is Hausdorff if and only if $\Delta = \{(y, y) \mid y \in Y\}$ is a closed subset of $Y \times Y$. (Indeed, that $\Delta \subseteq Y \times Y$ is closed means that each point $(x, y) \in Y \times Y \setminus \Delta$ admits a basic open neighborhood $(x, y) \in U \times V \subseteq Y \times Y \setminus \Delta$, i.e., $U, V \subseteq Y$ are disjoint open subsets with $x \in U$, $y \in V$.) Next, observe that $\Gamma(f)$ is the preimage of Δ under the continuous map $f \times 1 \colon X \times Y \to Y \times Y$.

(b) To show that f is continuous, we will show that $f^{-1}(C) \subseteq X$ is closed for all closed subsets $C \subseteq Y$. Note that $f^{-1}(C) = p\big((X \times C) \cap \Gamma(f)\big)$, where $p \colon X \times Y \to X$ is the projection. If C is closed in Y then $X \times C$ is closed in $X \times Y$ and hence $(X \times C) \cap \Gamma(f)$ is closed provided $\Gamma(f)$ is closed. Since Y is compact, p is a closed map, so $p\big((X \times C) \cap \Gamma(f)\big)$ is closed.

(3) (a) Suppose, to get a contradiction, that A is disconnected. This means that there is a surjective continuous map f from A to the discrete space $\{0, 1\}$. The restriction of f to $A \cap B$ must be constant since $A \cap B$ is connected. We may without loss of generality assume that $f(x) = 0$ for all $x \in A \cap B$. Now define $F \colon A \cup B \to \{0, 1\}$ by

$$F(x) = \begin{cases} f(x), & x \in A, \\ 0, & x \in B. \end{cases}$$

Since A and B are closed and since f agrees with the constant function with value 0 on the overlap $A \cap B$, the gluing lemma (Lemma 3.23 in Lee) guarantees that F is well-defined and continuous. But then F is a continuous surjective map from $A \cup B$ to $\{0, 1\}$, which contradicts connectedness of $A \cup B$. A similar argument shows B is connected.

(b) Take, for example, $X = \mathbb{R}$, $A = \{0, 1\}$, $B = (0, 1]$. Then $A \cup B = [0, 1]$ and $A \cap B = \{1\}$ are connected but A is not. Here B is not closed.

(4) (a) Define $p \colon \mathfrak{C}_2(S^2) \to S^2$ and $i \colon S^2 \to \mathfrak{C}_2(S^2)$ by $p(x, y) = x$ and $i(x) = (x, -x)$. Then $pi = 1$ and ip is homotopic to 1 via the homotopy $H \colon \mathfrak{C}_2(S^2) \times [0, 1] \to \mathfrak{C}_2(S^2)$ defined by

$$H\big((x, y), t\big) = \left(x, \frac{ty - (1-t)x}{|ty - (1-t)x|}\right).$$

To make sure H is well-defined one needs to check that $ty-(1-t)x \neq 0$ and $x \neq (ty-(1-t)x)/|ty-(1-t)x|$ for all $t \in [0,1]$. If $t=0$ or $y=-x$ one checks this by direct computation. If $t \neq 0$ and $y \neq -x$, it follows because either equality would imply that y is on the line spanned by x, which can not happen since x and $-x$ are the only two vectors of unit length on that line.

(b) First, note that the action of C_2 on $\mathfrak{C}_2(S^2)$ is a covering space action: For $(x,y) \in \mathfrak{C}_2(S^2)$, pick open sets $U, V \subseteq S^2$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$ (possible since S^2 is Hausdorff). Then $U \times V \subseteq \mathfrak{C}_2(S^2)$ is an open neighborhood of (x,y) and

$$(U \times V) \cap g(U \times V) = (U \times V) \cap (V \times U) = \emptyset,$$

for the non-trivial element g of C_2. Next, note that (a) implies that $\mathfrak{C}_2(S^2)$ is simply connected. It follows that

$$\pi_1\big(\mathfrak{C}_2(S^2)/C_2\big) \cong C_2.$$

(5) (a) The space X_n admits a polygonal presentation given by identifying all edges of an n-gon, oriented counterclockwise. This implies that $\pi_1(X_n)$ admits the presentation $\langle a \mid a^n \rangle$, i.e., $\pi_1(X_n)$ is a cyclic group of order n; we recall briefly the argument discussed in class: Let $U' = \{z \in D^2 \mid |z| < 1\}$ and $V' = \{z \in D^2 \mid |z| > 0\}$, and let $U = p(U')$ and $V = p(V')$, where $p\colon D^2 \to X_n$ is the quotient map. Then $X_n = U \cup V$ is an open cover and $U, V, U \cap V$ are path-connected. The diagram

$$\pi_1(U) \leftarrow \pi_1(U \cap V) \rightarrow \pi_1(V)$$

may be identified with $\{1\} \leftarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$, so the Seifert-Van Kampen theorem yields $\pi_1(X_n) \cong \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) \cong \mathbb{Z}/n\mathbb{Z}$. See Theorem 10.16 in Lee for a more detailed argument.

(b) By the classification of covering spaces, there is a one-to-one correspondence between isomorphism classes of covering spaces of X_n and conjugacy classes of subgroups of $\pi_1(X_n)$. All simply connected covers are isomorphic and correspond to the conjugacy class of the trivial subgroup. The trivial cover $X_n \to X_n$ corresponds to the conjuguacy class of the subgroup $\pi_1(X_n)$. Thus, the problem is equivalent to finding those n for which $\pi_1(X_n)$ has no other conjugacy classes of subgroups. Since $\pi_1(X_n)$ is cyclic of order n, this happens precisely when n is a prime number or $n = 1$, since the cyclic group of order n has a subgroup of order p for every prime divisor p of n, and a group of prime order has no subgroups apart from the trivial subgroup and the whole group.

(c) By the classification of surfaces, every compact surface is homeomorphic to S^2, $\#^g T^2$, or $\#^g \mathbb{R}\mathrm{P}^2$, for some $g \geq 1$. The abelianizations of the fundamental groups of these surfaces are given by $\pi_1(S^2)^{ab} \cong 0$, $\pi_1(\#^g T^2)^{ab} \cong \mathbb{Z}^{2g}$, and $\pi_1(\#^g \mathbb{R}\mathrm{P}^2)^{ab} \cong \mathbb{Z}^{g-1} \times \mathbb{Z}/2\mathbb{Z}$ (see Proposition 10.21 in Lee). The only finite cyclic groups appearing here are $\pi_1(S^2)^{ab} = 0$ and $\pi_1(\mathbb{R}\mathrm{P}^2)^{ab} = \mathbb{Z}/2\mathbb{Z}$. Since $\pi_1(X_n) \cong \pi_1(X_n)^{ab} \cong \mathbb{Z}/n\mathbb{Z}$ by (a), we can conclude that X_n is not a surface if $n \neq 1, 2$. The space X_1 is just the disk itself, which is not a surface since points on the boundary do not have euclidean neighborhoods (X_1 is however a surface with boundary). The space X_2 is obtained from the disk by identifying antipodal points on the boundary, so it is homeomorphic to $\mathbb{R}\mathrm{P}^2$. In summary, X_n is a surface if and only if $n = 2$.