Exam 02/08/2023 Solution

Exercise 1

(a) $C_2 = \sum_{k=0}^{1} C_k C_{1-k} = C_0 C_1 + C_1 C_0 = 2$ 0.5

$C_3 = \sum_{k=0}^{2} C_k C_{2-k} = C_0 C_2 + C_1 C_1 + C_2 C_0 =$
$= 2 + 1 + 2 = 5$ 0.5

(b) Let $G(x) = \sum_{k=0}^{\infty} C_k x^k$ 0.5

Using the product formula we have that

$$G(x)^2 = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} C_j C_{k-j} \right) x^k$$

$$= \sum_{k=0}^{m} C_{k+1} x^k \qquad 1$$

So we have that

$$x(G(x))^2 = \sum_{k=0}^{\infty} C_{k+1} x^{k+1}$$

$$= G(x) - C_0 \qquad 0.5$$

$$= G(x) - 1$$

We conclude that

$$x(G(x))^2 - G(x) + 1 = 0$$

(c) From B we have that

$$G(x)_{\pm} = \frac{1 \pm \sqrt{1-4x}}{2x}$$

We have to decide the ~~is~~ sign. We use the McLaurin expansion of $\sqrt{1-4x}$ to write $G(x)_{\pm}$ as a power series

$$G(x)_{\pm} = \frac{1 \pm (1-2x-2x^2+\cdots)}{2x}$$

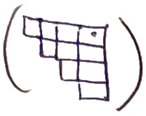
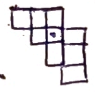
$$G(x)_{+} = \frac{2-2x-2x^2+\cdots}{2x}$$ not a power series (it does not converg if $x=0$)

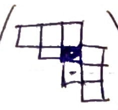
$$G(x)_{-} = \frac{\cancel{1}-\cancel{1}+2x+2x^2+\cdots}{2x}$$

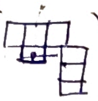
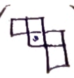
$$= 1 + 1x + \cdots$$

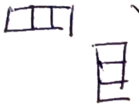
So we have that this is the right generating function

Exercise 2

$p($  $) = p($  $) + x\,p($  $)$

$= p($  $) + x\,p($  $) + x(1+3x+x^2)$

$= p($  $) + x\,p($  $) + x(1+2x)^2 + x+3x^2+x^3$

$= p($  $) + x\,p($  $) + x\,p($  $) + x^2 p($  $)$

$+ x + 4x^2 + 4x^3 + x + 3x^2 + x^3$

$$= (1+3x)^2 + x(1+2x)^2 + x(1+2x)^2 + x^2(1+x)^2$$
$$+ 2x + 7x^2 + 5x^3$$

$$= 1 + 6x + 9x^2 + 2x(1+4x+4x^2) + x^2 + 2x^3 + x^4$$
$$+ 2x + 7x^2 + 5x^3$$

$$= 1 + 8x + 17x^2 + 7x^3 + x^4 + 2x + 8x^2 + 8x^3$$

$$= 1 + 10x + 25x^2 + 15x^3 + x^4$$

Exercise 3

The characteristic polynomial is

$$x^2 + 4x + 4 = 0$$

$$\boxed{a_n = (-2)^n\left(-\frac{7}{8}n + \frac{3}{8}n^2\right)}$$

$$\Leftrightarrow \quad (x+2)^2 = 0$$

We have a real double root so ~~a char~~ the general solution of the homogeneous problem is

$$a_n^{(h)} = A(-2)^n + Bn(-2)^n$$

We guess a solution for the non homogeneous problem

$$a_n^{(p)} = Cn^2(-2)^n$$

$$C(n+2)^2(-2)^{n+2} + 4C(n+1)^2(-2)^{n+1} + 4Cn^2(-2)^n$$

$$= C(-2)^n\left(4(n^2+4n+4) - 8(n^2+2n+1) + 4n^2\right)$$

$$= C(-2)^n\left(\cancel{4n^2} + \cancel{16n} + 16 - \cancel{8n^2} - \cancel{16n} - 8 + \cancel{4n^2}\right)$$

$$= C\cancel{(-2)^n}\,\cancel{(\ldots)} \cdot 8 = 3\cancel{(-2)^n}$$

$$C \cdot 8 = 3 \qquad C = \frac{3}{8}$$

So the general solution of the ~~homogen~~ problem is

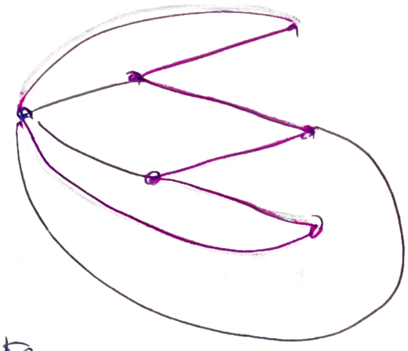
$$a_n = A(-2)^n + Bn(-2)^n + \frac{3}{8}n^2(-2)^n$$

$n=0$ $\quad 0 = a_0 = A$ $\qquad$ $n=1$ $\quad 1 = B(-2) + \frac{3}{8}(-2)$

$$B = -\frac{1}{2} - \frac{3}{8} = \frac{-4-3}{8} = -\frac{7}{8}$$

Exercise 4

(a)



(b) ~~Every vertex in the first par~~

$$|V(K_{100,100,100})| = 100+100+100$$

Let v is a vertex in the first partiton:
it is adjacent to every other vertices in the other partition.

$$\deg(v) = 100+100 = 200$$

the same is true for all the vertices in the other partition

$$|E| = \frac{1}{2}\sum_{v} \deg(v) = \frac{1}{2}\, 300 \cdot 200 = 30000$$

(c) if v is in partition i

$$\deg(v) = \sum_{j \neq i} n_j$$

So we want that

n_1+n_2
n_1+n_3 must be all even.
n_2+n_3

$\Longleftrightarrow$ n_1 n_2 & n_3 have the same parity.

(d) An Hamilton cycle can be seen in the picture

Otherwise we notice that the degree ~~of~~ ~~of~~ two non adjacent vertex is either 4 or 5

$|V(K_{1,2,3})| = 3+2+1 = 6$

$\deg v + \deg w \geq 4+4 = 8 > 6$

for any two non adjacent vertices

Exercise 5

We use Kruskal algorithm.

$\{e h\}$

$\{e f\}$ $\{h d\}$

$\{a\ b\}$

$\{e f\}$

$\{f g\}$

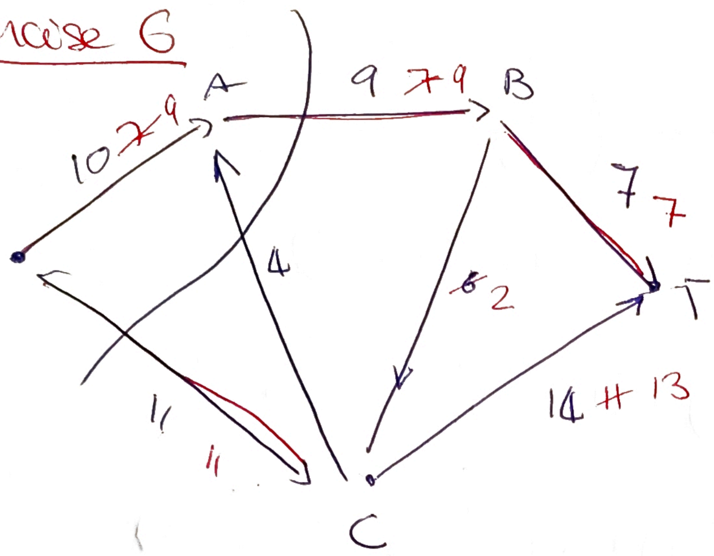
$\{b c\}$

$\{f b\}$

$\text{Weight} = 1+2+4+5+5+6+7$

$= 34$

Exercise 6



The max flow is $9+11 = 13+7 = 20$

The cut $\{S\,A\}$ $\{B\,C\,T\}$ has the same capacity

$$C(P, P^c) = w(AB) + w(SC)$$
$$= 9 + 11 = 20$$