MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Tentamensskrivning i
Mathematics Methods for Economist MM3001
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Instructions:

- During the exam you MAY NOT use textbooks, class notes, or any other supporting material apart from the formula sheet given to you.
- Use of calculators is permitted for performing calculations. The use of graphic or programmable features is NOT permitted.
- You can use the formula sheet that come with the exam.
- Start every problem on a new page, and write at the top of the page which problem it belongs to. (But in multiple part problems it is not necessary to start every part on a new page)
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for unclear and wrong argument, even if the final answer is correct.
- Write clearly and legibly.
- Where applicable, indicate your final answer clearly by putting A BOX around it.

Note: There are six problems, some with multiple parts. The problems are not ordered according to difficulty

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(1) (5pt) Compute the degree 3 Taylor polynomial of the function

$$f(x) = (x-1)\ln(x^2+1),$$

around the point $x_0 = 0$, and use it to give an approximation of $f(0.1)$.
Solution We have that $f(0) = 0$. We comput the first derivative of f

$$\begin{aligned} f'(x) &= \ln(x^2+1) + (x-1)2x\frac{1}{x^2+1} \\ &= \ln(x^2+1) + \frac{2x^2-2x}{x^2+1} \end{aligned}$$

Thus $f'(0) = 0$. We compute the second derivative.

$$\begin{aligned} f''(x) &= \frac{2x}{x^2+1} + \frac{(4x-2)(x^2+1)-(2x^2-2x)(2x)}{(x^2+1)^2} \\ &= \frac{2x}{x^2+1} + \frac{4x-2+4x^3-2x^2-4x^3+4x^2}{(x^2+1)^2} \\ &= \frac{2x^3+2x+4x-2+2x^2}{(x^2+1)^2} \\ &= \frac{2x^3+2x^2+6x-2}{(x^2+1)^2} \end{aligned}$$

Thus $f''(0) = -2$. We compute the third derivative

$$f^{(3)}(x) = \frac{(6x^2+4x+6)(x^2+1)^2 - (-2x^3+4x^2+6x-2)2x2(x^2+1)}{(x^2+1)^4}$$

We have that $f^{(3)}(0) = 6$. We conclude that the Taylor polynomial of the function around $x_0 = 0$ is

$$\begin{aligned} p(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 \\ &= -x^2 + x^3 = x^2(x-1). \end{aligned}$$

We compute $p(0.1)$.

$$p(0.1) = -\frac{1}{100} + \frac{1}{1000} = \frac{-9}{1000}$$

(2) Geometric Series: Consider the following sequence:

$$a_0 = 3, \quad a_1 = \frac{3}{(1+2p)^2}, \quad a_2 = \frac{3}{(1+2p)^4}, \quad a_3 = \frac{3}{(1+2p)^6}, \ldots$$

(a) (2 pt) Show that a_n determines a geometric progression, compute the geometric ratio and give an expression for

$$\sum_{n=2}^{6} a_n$$

(b) (2pt) Determine for which value of p the infinite series

$$S(p) = \sum_{n=0}^{\infty} 3(1+2p)^{-2n}$$

converges.

(c) (1pt) Determine if there is a p such that $S(p) = \frac{7}{4}$.

Solution (a) We have that

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{\frac{3}{(1+2p)^{2(n+1)}}}{\frac{3}{(1+2p)^{2n}}} \\ &= \frac{3(1+2p)^{2n}}{2(1+2p)^{2(n+1)}} = \frac{1}{(1+2p)^2}. \end{aligned}$$

This does not depend of n, thus the progression is geometric. The geometric ratio is $\frac{1}{(1+2p)^2}$ and the coefficient is 3.

We have that

$$\begin{aligned} \sum_{n=2}^{6} a_n &= 3\sum_{n=2}^{6} \frac{1}{(1+2p)^{2n}} \\ &= 3\frac{1}{(1+2p)^4}\sum_{n=0}^{4} \frac{1}{(1+2p)^{2n}} \\ &= 3\frac{1}{(1+2p)^4}\left(\frac{1-\left(\frac{1}{(1+2p)^2}\right)^5}{1-\frac{1}{(1+2p)^2}}\right) \end{aligned}$$

(b) For the series to converge we have to have that the absolute value of the geometric ratio is smaller than 1, that is

$$\left|\frac{1}{(1+2p)^2}\right| < 1.$$

As squares cannot be negative we have that this is equivalent to

$$\frac{1}{(1+2p)^2} < 1.$$

With the aid of elementary algebra we get that this is equivalent to

$$0 < 4p(1+p)$$

which is true whenever $p \notin [-1, 0]$, or, equivalently, whenever $p < -1$ or $p > 0$. (c) When $p \notin [-1, 0]$ we have that

$$S(p) = \frac{3}{1 - \frac{1}{(1+2p)^2}} = \frac{3(1+2p)^2}{(1+2p)^2 - 1} = \frac{3 + 12p + 12p^2}{4p + 4p^2}.$$

We want to investigate if the equation $S(p) = \frac{7}{4}$ has solutions this is equivalent to the equation

$$\frac{3 + 12p + 12p^2}{4p + 4p^2} = \frac{7}{4}.$$

By performing elementary algebra operations we get that this is equivalent to

$$5p^2 + 5p + 3 = 0$$

which has no solution. So the equation $S(p) = \frac{7}{4}$ has no solution.

(3) Consider the function $f(x) = \frac{x^2-9}{x^2-4}$.
  (a) (1pt) Find the natural domain of $f(x)$ and the solutions to $f(x) = 0$.
  (b) (2pt) Find where the function is increasing or decreasing. Find the critical points of $f(x)$ and determine their type.
  (c) (1pt) Find the max and min value of the function on the interval $[-1, 1]$.
  (d) (1pt) Compute $\lim_{x\to\pm\infty} f(x)$ and sketch the graph of f.

Solutions (a)The natural domain of the function is $x \neq \pm 2$ and the function is 0 when $x = \pm 3$. (b) We have to compute the first derivative of f. We have

$$f'(x) = \frac{2x(x^2-4) - 2x(x^2-9)}{(x^2-4)^2} = \frac{10x}{(x^2-4)^2}$$

We have that $f'(x) = 0$ when $x = 0$, in addition $f'(x)$ is positive when x is positive and $f'(x)$ is negative when x is negative. Thus we have the following table

		-3		-2		0		2		3	
$f'(x)$	-	-	-	?	-	0	+	?	+	+	+
$f(x)$	↘	0	↘	?	↘	local min	↗	?	↗	0	↗

The function is increasing when $x > 0$ and decreasing when $x < 0$. There is just one critical point in $x = 0$, which is a local min. (b) The candidates for extreme points in a closed interval are, the extremes of the interval, the critical points lying in the interior of the interval and the points in the interior of the interval where the function is not differentiable. The function f is always differentiable in its domain so we only have -1, 1 and 0. We compare the values the functions takes in these points we have

x	$f(x)$
-1	$\frac{8}{3}$
0	$\frac{9}{4}$
1	$\frac{8}{3}$

As $9/4 < 8/3$ we have that -1 and 1 are global max points for the function in $[-1, 1]$ and 0 is a global min point. The maximum value of the function in $[-1, 1]$ is $8/3$, and the minimum value i $9/4$.

(4) Compute the following integrals:

(a) (2.5 pt) $\int \left(\frac{\ln(y)}{((\ln(y))^2+1)} \frac{1}{y} + \sqrt[3]{y^8} \right) dy$,

(b) (2.5pt) $\int_0^1 x e^{x+1} dx$.

Solution (a) We split our integral in the sum of two integrals

$$\int \left(\frac{\ln(y)}{((\ln(y))^2+1)} \frac{1}{y} + \sqrt[3]{y^8} \right) dy = \int \frac{\ln(y)}{((\ln(y))^2+1)} \frac{1}{y} dy + \int \sqrt[3]{y^8} dy$$

We begin with the second summand

$$\int \sqrt[3]{y^8} dy = \int y^{8/3} dy = \frac{1}{1+\frac{8}{3}} y^{\frac{8}{3}+1} + C_1 = \frac{3}{11} y^{11/3} + C_1.$$

Let us integrate the first summand

(1)
$$\int \frac{\ln(y)}{((\ln(y))^2+1)} \frac{1}{y} dy = \int \frac{z}{z^2+1} dz$$

(2)
$$= \frac{1}{2} \int \frac{2z}{z^2+1} dz$$

(3)
$$= \frac{1}{2} \int \frac{1}{u} du$$

(4)
$$= \frac{1}{2} \ln|u| + C_2 = \frac{1}{2} \ln|z^2+1| + C_2 = \frac{1}{2} \ln|ln(y)^2+1| + C_2.$$

Where in (2) we have made the substitution $\ln(y) = z$ and in (4) we have substituted $z^2+1 = u$. In alternative one could have solved this just with one substitution $u = \ln(y)^2 + 1$.

In conclusion we have

$$\int \left(\frac{\ln(y)}{((\ln(y))^2+1)} \frac{1}{y} + \sqrt[3]{y^8} \right) dy = \frac{1}{2} \ln|ln(y)^2+1| + \frac{3}{11} y^{11/3} + C$$

(b) We observed that the function is defined in the integration interval. We substitute $z = x+1$ and we get

$$\int_0^1 x e^{x+1} dx = \int_1^2 (z-1) e^z dz = \int_1^2 z e^z dz - \int_1^2 e^z dz$$

The second integral is

$$\int_1^2 e^z dz = [e^z]_1^2 = e^2 - e.$$

We solve the first integral by parts

$$\int_1^2 z e^z dz = [z e^z]_1^2 - \int_1^2 e^z = 2e^2 - e - (e^2 - e) = e^2.$$

Putting all together we have

$$\int_0^1 x e^{x+1} dx = e^2 - e^2 + e = e$$

(5) Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 4+c \\ -4 & 1 & 0 \\ c & 0 & -2 \end{pmatrix}$$

(a) (2 pt) Compute the determinant of A, $|A|$ as a function of c.

(b) (1 pt) Find all the values of c for which A is not invertible.
(c) (2 pt) Find how many solutions has the following linear system:

$$\begin{cases} 2x & -y & +5z & = & 4 \\ -4x & +y & & = & 13 \\ x & & -2z & = & -8 \\ 5x & -y & -2z & = & -21 \end{cases}$$

Solution (a)

$$\begin{aligned} |A| = & \begin{vmatrix} 2 & -1 & 4+c \\ -2 & 0 & 4+c \\ c & 0 & -2 \end{vmatrix} \\ & = (-1)^{1+2}((-2)(-2) - c(4+c)) \\ & = -(4 - 4c - c^2) \end{aligned}$$

(b) We have that A is invertible whn its determinant is nonzero. Now the equation

$$c^2 + 4c - 4 = 0$$

has two solutions $c_{\pm} = -2 \pm \sqrt{8}$ which means that A is invertible whenever $c \neq -2 \pm \sqrt{8}$. (c) the augmented matrix associated to the linear system is

$$\left(\begin{array}{ccc|c} 2 & -1 & 5 & 4 \\ -4 & 1 & 0 & 13 \\ 1 & 0 & -2 & -8 \\ 5 & -1 & -2 & -21 \end{array}\right)$$

We run Gauss–Jordan Elimination;

$$\begin{aligned} \left(\begin{array}{ccc|c} 2 & -1 & 5 & 4 \\ -4 & 1 & 0 & 13 \\ 1 & 0 & -2 & -8 \\ 5 & -1 & -2 & -21 \end{array}\right) & \sim \left(\begin{array}{ccc|c} 2 & -1 & 5 & 4 \\ -2 & 0 & 5 & 17 \\ 1 & 0 & -2 & -8 \\ 3 & 0 & -7 & -25 \end{array}\right) \\ & \sim \left(\begin{array}{ccc|c} 2 & -1 & 5 & 4 \\ 1 & 0 & -2 & -8 \\ -2 & 0 & 5 & 17 \\ 3 & 0 & -7 & -25 \end{array}\right) \\ & \sim \left(\begin{array}{ccc|c} 2 & -1 & 5 & 4 \\ 1 & 0 & -2 & -8 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \\ & \sim \left(\begin{array}{ccc|c} 2 & -1 & 5 & 4 \\ 1 & 0 & -2 & -8 \\ 0 & -0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{aligned}$$

By looking at the last row it gives an equation 0=0. Thus the system has either 1 solution or infinitely many solutions. We have that there are 3 variables and 3 non-zero rows, so, the systema has only one solution.

(6) Consider the two variables function

$$f(x, y) = y^3 + 2x^2 + 4x - 27y + 100$$

defined on the square

$$D = \{(x, y) \mid -3 \leq x \leq 0,\ -3 \leq y \leq 0\}$$

(a) (2pt) Find all the critical points of $f(x, y)$ - even those lying outside D and determine their type.
(b) (2pt) Determine the maximum and minimum points of f on the *boundary* of D. (In order to get credit you have to explain what you are doing, the correct answer without the right explanation will not be accepted)
(c) (1 pt) Determine the minimum and the maximum value of $f(x, y)$ on D. (In order to get credit you have to explain what you are doing, the correct answer without the right explanation will not be accepted)

Solution: We compute the first order partial derivative and set them to 0

$$\frac{\partial}{\partial x} f(x, y) = 4x + 4 = 0$$
$$\frac{\partial}{\partial y} f(x, y) = 3y^2 - 27 = 0$$

The system has only two solutions $(-1, 3)$ and $(-1, -3)$. To classify the type of the two critical points we need to compute the Hessian

$$H(x, y) = \begin{vmatrix} 4 & 0 \\ 0 & 6y \end{vmatrix} = 24y$$

The Hessian is positive, when y is. So if $(x, y) = (-1, -3)$ we have a saddle point. If $(x, y) = (-1, 3)$, since the unmixed derivatives $\partial_{xx} f(x, y)$ and $\partial_{yy}(x, y)$ are positive, the two critical point is a local minimum.
(b) The boundary of the square is consisting in 4 sides.
Side 1: $x = 0$, $y \in [-3, 0]$. Let $g(y) = f(0, y) = y^3 - 27y + 100$. We want to find the min and max value of $g(y)$ when $y \in [0, 3]$. T0 this aim we compute $g'(y) = 3y^2 - 27$ which has two roots $y = -3$ and $y = +3$. Non of these lie in the interval we are considering, So the only candidateds for max and min on this side are $(0, -3)$ and $(0, 0)$.
Side 2: $y = 0$, $x \in [-3, 0]$. Let $g(x) = f(x, 0) = 2x^2 + 4x + 100$. We want to find the min and max value of $g(x)$ when $x \in [0, 3]$. To this aim we compute $g'(x) = 4x + 4$ which has one root, $x = -1$ which lies in the interval. Non of these lie in the interval we are considering, So the only candidates for max and min on this side are $(-3, 0)$ and $(0, 0)$ and $(-1, 0)$.
Side 3: $x = -3$, $y \in [-3, 0]$. Let $g(y) = f(-3, y) = 18+y^3-12-27y+100 = y^3 - 27y + 106$. We want to find the min and max value of $g(y)$ when $y \in [0, 3]$. T0 this aim we compute $g'(y) = 3y^2 - 27$ which has two roots $y = -3$ and $y = +3$. Non of these lie in the interval we are considering, So the only candidates for max and min on this side are $(-3, -3)$ and $(-3, 0)$.
Side 4: $y = -3$, $x \in [-3, 0]$. Let $g(x) = f(x, 0) = -27+2x^2+4x+81+100 = 2x^2 + 4x + 154$. We want to find the min and max value of $g(x)$ when $x \in [0, 3]$. To this aim we compute $g'(x) = 4x + 4$ which has one root, $x = -1$ which lies in the interval. Non of these lie in the interval we are considering, So the only candidates for max and min on this side are $(-3, -3)$ and $(0, -3)$ and $(-1, -3)$.
Summarizing we have the following table

(x,y)	$f(x,y)$	
(-3,0)	106	
(0,0)	100	
(0,-3)	154	
(-3,-3)	159	this is the max
(-1,-3)	152	
(-1,0)	98	this is the min

Thus the maximum value of $f(x,y)$ on the boundary of D is 159 - taken in $(-3,-3)$- and the minimum is 98 - taken in $(-1,0)$.
(c) Of the critical points found in (a) non lies in the interior of the boundary. We conclude that the max and min value of the function on D are the same as those on the boundary. Thus the maximum value of $f(x,y)$ on D is 159 - taken in $(-3,-3)$- and the minimum is 98 - taken in $(-1,0)$.

GOOD LUCK!!!