MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Yishao Zhou

Tentamensskrivning i
MM7027, 7,5 hp
March 17, 2023

You are allowed to bring an A-4 sheet (double sides) with whatever you think is important.
Write down on your cover page how many points you collected from the homework and project assignments.
You must motivate well your arguments.

Through out this exam we assume $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $b, c' \in \mathbb{R}^n$, unless the positive integers n, m, p are apparently concrete numbers as for example in 5(i).

1. Consider the two dynamical systems

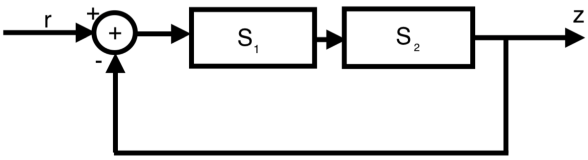
$$
\begin{aligned}
(S_1) \quad & \dot{x}_1 = x_2 + u,\ \dot{x}_2 = -2x_1 - 3x_2,\ y = \alpha x_1 + x_2, \\
(S_2) \quad & \dot{x}_3 = x_3 + w,\ z = x_3,
\end{aligned}
$$

where (S_1) has state (x_1, x_2), control u and output y, (S_2) has state x_3, control w and output z, and α is a real parameter.

(i) Determine whether each system is controllable, observable, stable.

(ii) These two systems are connected in series with $w = y$. Call the resulting system (S_3). Determine whether it is controllable, observable, stable.

(iii) The systems are now connected in a feedback configuration as shown below to produce (S_4). Determine whether it is controllable, observable, stable.



12 p

2. Show that if there are $n \times n$ symmetric positive definite matrices P and Q such that the $n \times n$ matrix equation $PA + A'P + 2\lambda P = -Q$ holds, where λ is a real number, then all eigenvalues of A have a real part that is less than $-\lambda$. 12 p

3. Show that if the system $\dot{x}(t) = Ax(t) + Bu(t)$ is completely controllable then the system $\dot{x}(t) = Ax(t) + BKx(t) + Bu(t)$ is also completely controllable. 12 p

4. Find the control that transfers $\dot{x}(t) = b(t)u(t)$ from the state $x(0) = 1$ to the state $x(1) = 0$ and minimizes $\eta = \int_0^1 u^2 dt$. 12 p

5. (i) Show that if (A, b, c) is a realization such that $c[\mathrm{Adj}(sI - A))]b = s^2 + 3s + 2$ and $\det(sI - A) = s^3 + 3s^2 - s - 3$, then the realization is not minimal.

(ii) Show that A and bc do not commute if the realization is minimal.

12 p

You have finished the exam if the point of your homework is $p_h \geq 24$ and your project part is graded as Pass. Continue otherwise. Now if you don't collected enough points, follow carefully the instructions below.

6. (*If you passed the project, go to next.*)

Consider the linear time-invaraint system $\dot{x} = Ax + Bu, y = Cx$.

(i) Why is direct output feedback alone in general not sufficient?

(ii) Explain the ideas of (state) observer and provide a dynamic feedback to the given system.

10 p

You have finished the exam if the point of your homework is $p_h \geq 24$. Continue otherwise.

7. Assume that (A, b) is controllable. Let c_n' be the last row of the inverse of the controllability matrix $M = \begin{pmatrix} b & Ab & \cdots & A^{n-1}b \end{pmatrix}$.

(i) Show that the matrix $P = \begin{pmatrix} c_n' \\ c_n' A \\ \vdots \\ c_n' A^{n-1} \end{pmatrix}$ is invertible.

(ii) Determine the matrix $\tilde{A} = PAP^{-1}$ and the vector Pb. Show that $A^n b$ depends only on the matrix A.

(iii) Show that there is a unique n-vector k such that $A - bk^t$ has all its eigenvalues with negative real parts.

10 p

You have finished the exam if your homework is $23 \geq p_h \geq 16$. Continue otherwise.

8. Assume that the linear system $\dot{x}(t) = Ax(t)$ is asymptotically stable.

(i) Show that the corresponding discrete-time system defined by

$$\frac{x(k+1) - x(k)}{\Delta} = Ax(k+1)$$

is also asymptotically stable, where Δ is the step size.

(ii) Find the condition on the step size Δ under which $x(k)$ obtained by

$$\frac{x(k+1) - x(k)}{\Delta} = Ax(k)$$

converges to the solution of the equation $\dot{x}(t) = Ax(t)$.

10 p

You have finished the exam if your homework is $16 \geq p_h \geq 8$. Continue otherwise.

9. Consider the algebraic Riccati equation (ARE) $KA + A'K - KBB'K + C'C = 0$.

(i) Show that if (A, C) is observable then every solution of the ARE is invertible.

(ii) Assume that K_∞ solves the ARE such that the real parts of all eigenvalues of $A - BB'K_\infty$ are negative and that $P = K - K_\infty$ is invertible. Find the solution of the ARE in a closed form.

(iii) Show that the ARE cannot be generically quadratic if (A, B) is not controllable, that is, it can be decomposed to an algebraic Riccati equation of smaller size which cannot be decomposed further and two (matrix) Lyapunov equations.

10 p

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