Solutions to the exam 230317, MM7027

1. (i) System (S_1) is completely controllable, asymptotically stable and completely observable if $\alpha \neq 2$ and $\alpha \neq 1$, since the characteristic polynomial of the matrix $A_1 = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ is $s^2 + 3s + 2$ has positive coefficients which implies that the eigenvalues have negative real part, and the controllability matrix $(b_1, A_1 b_1) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ has rank 2 and the observability matrix $\begin{pmatrix} c_1 \\ c_1 A_1 \end{pmatrix} = \begin{pmatrix} \alpha & 1 \\ -2 & \alpha - 3 \end{pmatrix}$ which has rank 2 if and only if $\alpha \neq 2$ and $\alpha \neq 1$.
System (S_2) is unstable since $A_2 = 1 > 0$. It is completely controllable since $b_2 = 1 \neq 0$ which means the controllability matrix has rank 1 and completely observable because $c_2 = 1 \neq 0$ implying the observability matrix has rank 1.

   (ii) System (S_3) is in the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -2 & -3 & 0 \\ \alpha & 1 & 1 \end{pmatrix}}_{A_3} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{b_3} u,, \quad z = \underbrace{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}}_{c_3} x.$$

So the system is unstable since there is a positive real eigenvalue at 1. System (S_3) is completely controllable if $\alpha \neq 1/2$ since the rank of the controllability matrix

$$(b_3 \quad A_3 b_3 \quad A_3^2 b_3) = \begin{pmatrix} 1 & 0 & -2 \\ 0 & -2 & 6 \\ 0 & \alpha & \alpha - 2 \end{pmatrix}$$

is 3 if $\alpha \neq 1/2$. It is completely observable if $\alpha \neq 1$ and $\alpha \neq 2$ since under this condition the observability matrix

$$\begin{pmatrix} c_3 \\ c_3 A_3 \\ c_3 A_3^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \alpha & 1 & 1 \\ \alpha - 2 & \alpha - 2 & 1 \end{pmatrix}$$

has rank 3.

   (iii) Note that the input fior (S_1) is $r - z = r - x_3$ and the input for (S_2) is $w = y = \alpha x_1 + x_2$ So the feedback system S (S_4) is governed by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & -1 \\ -2 & -3 & 0 \\ \alpha & 1 & 1 \end{pmatrix}}_{A_4} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{b_4} r,, \quad z = \underbrace{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}}_{c_4} x.$$

Now $\chi_{A_4}(s) = s^3 + 2s^2 + (\alpha - 1)s + 3\alpha - 4$. It has all zeros with negative real part if and only if $\alpha - 1 > 0$, $3\alpha - 4 > 0$ and $2(\alpha - 1) - (3\alpha - 4) > 0$, i.e. $\frac{4}{3} < \alpha < 2$. That is the system is asymptotically stable if $\frac{4}{3} < \alpha < 2$. It is completely controllable if $\alpha \neq 1/2$ since

$$(b_4 \quad A_4 b_4 \quad A_4^2 b_4) = \begin{pmatrix} 1 & 0 & -2 - \alpha \\ 0 & -2 & 6 \\ 0 & \alpha & \alpha - 2 \end{pmatrix}$$

has rank 3. It is completely observable if $\alpha \neq 1$ and $\alpha \neq 2$, since then

$$\begin{pmatrix} c_4 \\ c_4 A_4 \\ c_4 A_4^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 - \alpha \\ \alpha & 1 & 1 \\ \alpha - 2 & \alpha - 2 & -\alpha + 1 \end{pmatrix}$$

has rank 3.

2. Note that the equation is equivalent to $P(\underbrace{A + \lambda I}_{\tilde{A}}) + (A + \lambda I)'P = -Q$. Let μ be any eigenvalue of $\tilde{A}$ and $v \neq 0$ its associate eigenvector., i.e. $\tilde{A}v = \mu v$. Multilplying the above equation v^* from left and v from right yields $0 > -v^*Qv = (\bar{\mu} + \mu)v^*Pv$ which is $\bar{\mu} + \mu) < 0$ since P, Q are symmetric positive definite. This means that the real part of $\mu < 0$. Note that the eigenvalues of $\tilde{A}$ are λ plus the eigenvalues of A so the real part of eigenvalues of A must be less than $-\lambda$.

3. By the Hautus Lemma, $(\lambda I - A, B)$ has full rank, n, for any $\lambda \in \mathbb{C}$. Now
$$(\lambda I - (A + BK), B) = (\lambda I - A, B) \begin{pmatrix} I & 0 \\ -K & I \end{pmatrix}$$
and $\begin{pmatrix} I & 0 \\ -K & I \end{pmatrix}$ is non-singular. So the rank of $(\lambda I - (A+BK), B)$ is equal to the rank of $\lambda I - A, B)$ which is n for any $\lambda \in \mathbb{C}$, showing that $(A + BK, B)$ is controllable by the Hautus Lemma again.

4. By the boundary conditions for x we see that $b \neq 0$. So the reachability grammian $\int_0^1 b(t)^2 dt \neq 0$. So the system can be driven by the choice of u to the state $x(1) = 0$ from $x(0) = 1$. So we have the Hamiltonian
$$H(p, x, u) = u^2 + pbu.$$
where $p \neq 0$ by the Pontragin Minimum Principle. Clearly
$$u^*(t) = \arg\min_u H(p, x, u) = \arg\min_u \left(u + \frac{pb}{2}\right)^2 - \frac{p^2 b^2}{4} = -\frac{pb}{2}.$$
Hence $H(p, x, u^*) = -\frac{b^2 p^2}{4}$. The function p satisfies $\dot{p} = 0$ without boundary conditions. $p(t) = C$, a constant. Now we solve x from $\dot{x}^* = b(t)u^*(t) = -b^2(t)\frac{C}{2}$. Integrating this equation we get $x^*(t) = 1 - \frac{C}{2}\int_0^t b^2(s)ds$. Using $x^*(1) = 0$ yields $C = 2/\int_0^1 b^2(t)dt$. To summarize, the optimal control $u^*(t) = b(t)/\int_0^1 b^2(s)ds$ which transfers $\dot{x} = bu$ from $x(0) = 1$ to $x(1) = 1$.

5. (i) The realization under the given condition is three dimension. Since $s + 1$ is a common factor of the given polynomials $c(sI - A)^{-1}b = \frac{s^2+3s+2}{s^3+3s^2-s-3} = \frac{s+2}{s^2+2s-3}$. So we have a realization of dimension 2, proving that the realization is not minimal.
(ii) Consider the minimal realization in controller form
$$A^\flat = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}, \; b^\flat = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}, \; c^\flat = (c_1, ..., c_n)$$
A straightforward computation show that $A^\flat(b^\flat c^\flat) \neq (b^\flat c^\flat)A^\flat$. For any given minimal realization (A, b, c) of dimension n it is similar to $(A^\flat, b^\flat c^\flat)$, i.e. there is a nonsingular matrix P such that
$$A^\flat = P^{-1}AP, \; b^\flat = P^{-1}b, \; c^\flat = cP.$$
Clearly $A(bc) = (PA^\flat P^{-1})((Pb^\flat)(c^\flat P^{-1})) = P(A^\flat(b^\flat c^\flat))P^{-1}$ and $(bc)A = ((Pb^\flat)(c^\flat P^{-1}))(PA^\flat P^{-1}) = P(b^\flat c^\flat)A^\flat P^{-1}$ are not equal. Hence if (A, b, c) is minimal realization then A and bc do not commute.

6. See lecture notes `https://kurser.math.su.se/pluginfile.php/161815/mod_resource/content/0/Day8_Dynamic_feedback.pdf`

7. see lecture notes `https://kurser.math.su.se/pluginfile.php/161350/mod_resource/content/0/Day5_controllabilty_tests_pole-shifting.pdf`

8. Let $\lambda_1, ..., \lambda_n$, counted with multiplicity, be eigenvalues of A. Since $\dot{x} = Ax$ is asymptotically stable, $\lambda_i \in \mathbb{C}^-$. Note also that the stepsize $\Delta > 0$.
(i) The iteration is $x(k + 1) = (I - \Delta A)^{-1}x(k)$. It converges if and only if all the eigenvalues of $I - \Delta A$, $1 - \Delta\lambda_i$, $i = 1, ..., n$ lie outside of the unite circle, or equivalently, $\Delta\lambda_i$ lies outside of the circle $\{z : |z - 1| = 1\}$ which covers $\mathbb{C}^-$ for all $\Delta > 0$. So it converges to the solution of $\dot{x} = Ax$ which is assumed to be asymptotically stable.
(ii) In this case $x(k+1) = (I + \Delta A)x(k)$. It converges if and only if the eigenvalues of $I + \Delta A$ lies inside the unit circle of $\Delta\lambda_i$ lies in open the disk $\{z : |z + 1| < 1\} \subset \mathbb{C}^-$. In order to have $x(k)$ converge to the solution of $\dot{x} = Ax$, we have to impose the condition on the stepsize Δ: $|1+\Delta\lambda_i| < 1$, for alla $i = 1, ..., n$, which holds true if $\Delta < 2/\max_i |\lambda_i|$.

9. (i) See `https://kurser.math.su.se/pluginfile.php/162600/mod_resource/content/3/final_review.pdf` on page 4.

(ii) See `https://kurser.math.su.se/pluginfile.php/162600/mod_resource/content/3/final_review.pdf` on page 6.

(iii) Assume the controllability subspace has dimension $r < n$. By the Kalman decomposition there is a nonsingular matrix P such that

$$\tilde{A} = P^{-1}AP = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \; \tilde{B} = P^{-1}B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \; \tilde{C} = CP = (C_1, C_2),$$

where (A_1, B_1) is controllable. Using these relation and multiply the ARE P^{-1} from left and P from right yield

$$K_{11}A_1 + A_1'K_{11} - K_{11}B_1B_1' + C_1'C_1 = 0,$$

which is an ARE of $r \times r$ matrix equation, where $\tilde{K} = P^{-1}KP = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}$, and

$$K_{12}A_1 + A_2'K_{11} + A_3'K_{12}' + K_{12}B_1B_1'K_{11} + C_2'C_1 = 0,$$

which is Lyapunov equations for K_{12} if the r-dimensional ARE is solved for K_{11}, and finally

$$K_{12}A_2 + K_{22}A_3 + A_2'K_{12} + A_3'K_{22} + K_{12}B_1B_1'K_{12} + C_2'C_2 = 0$$

another Lyapunov equation but for K_{22}.