Department of Mathematics
Stockholm University

MA 5020 - Abstract Algebra

Spring 2023 - final exam

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- **No** use of textbook, notes, or calculators is allowed.
- Unless told otherwise, you may quote results that were proved in class. When you do, state precisely the result that you are using.
- Be sure to justify your answers, and show clearly all steps of your solutions.
- In problems with multiple parts, results of earlier parts can be used in the solution of later parts, even if you do not solve the earlier parts

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1. Let G be a group, and $N \triangleleft G$ a normal subgroup. For each of the following statements, determine if it is true or false. Give a brief justification or a counterexample.
   (a) (2 points) If N and G/N are both abelian, then G is abelian.
   (b) (2 points) If G is abelian then N and G/N are both abelian.
2. Let G be a group and $H_1, H_2 \subset G$ two subgroups. Recall that H_1H_2 is the set of all products $\{h_1h_2 \in G \mid h_1 \in H_1, h_2 \in H_2\}$.
   (a) (3 points) Show an example where H_1H_2 is not a subgroup of G.
   (b) (3 points) Prove that if $H_1H_2 \subseteq H_2H_1$ then in fact $H_1H_2 = H_2H_1$.
3. Suppose G is a group that acts transitively on the left on a set X. Recall that "transitively" means that for every two elements $x_1, x_2 \in X$ there exists a $g \in G$ such that $gx_1 = x_2$.
   (a) (3 points) Let $x_1, x_2 \in X$. Prove that the stabilizers of x_1 and x_2 are conjugate subgroups of G.
   (b) (2 points) Suppose in addition that G is finite. Prove that there exists an element $g \in G$ that satisfies $gx \neq x$ for all $x \in X$ (in other words, prove that there exists an element of G that does not fix any element of X).
4. (a) (2 points) Prove that a group of order 56 can not be simple.
   (b) (3 points) Prove that a group of order 72 can not be simple.
5. (a) (2 points) Let R be a ring, and I, J ideals of R. Suppose that $I \cap J$ is a prime ideal of R. Prove that either $I \subseteq J$ or $J \subseteq I$.
   (b) (3 points) Let R be a ring that satisfies $x^2 = x$ for all $x \in R$. Prove that R is commutative.
6. Let $\mathbb{F}$ be a field, and $\mathbb{F}[x]$ the ring of polynomials over $\mathbb{F}$. In this question you will consider the ideals (x^2+1) and (x^2-1) in $\mathbb{F}[x]$, and the quotient rings $\mathbb{F}[x]/(x^2+1)$ and $\mathbb{F}[x](x^2-1)$. Be sure to justify your answers.
   (a) (2 points) Suppose $\mathbb{F} = \mathbb{C}$ is the field of complex numbers. Are the rings $\mathbb{C}[x]/(x^2+1)$ and $\mathbb{C}[x]/(x^2-1)$ isomorphic?
   (b) (3 points) Suppose $\mathbb{F} = \mathbb{F}_3$ is the field with three elements. Are the rings $\mathbb{F}_3[x]/(x^2+1)$ and $\mathbb{F}_3/(x^2-1)$ isomorphic?