MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Written exam in
Advanced Real Analysis II,
MM8039 (SF2744)
May 24, 2017
9:00-14:00

30 *points can be obtained from the written exam* $((3+3) \times 4$ *including bonus) and the oral exam* $(3+3)$.

Credit scale:

A= at least 26,5 points,	*B= at least 23 points,*	*C= at least 20 points,*
D= at least 17,5 points,	*E= at least 15 points*	*Fx= at least 13,5 points.*

Important: *If you intend to take the oral exam, email to luger@math.su.se no later than Friday 26/5.*

Motivate your solutions carefully!!!

I-1 Let $E \subset \mathbf{R}$, and μ the Lebesgue measure, with the property that

$$\mu(I_r(z) \setminus E) \geq \frac{r}{100}, \qquad \forall\, z \in E, \qquad \forall\, r > 0,$$

where $I_r(z) = (z, r+z)$.

Prove that $\mu(E) = 0$.

Hint: Think of indefinit integrals, and that $g' = f$ *a.e., when* g *is primitive of* f*, for integrable* f.

I-2 State and prove Lebesgue decomposition theorem.

I-3 Let $m > 0$, and define

$$E_0 := \{0\} \cup \left\{1, \frac{1}{2^m}, \frac{1}{3^m}, \frac{1}{4^m}, \cdots\right\}, \qquad \text{and} \qquad E_1 = E_0 \times [0,1].$$

Find the upper Minkowski dimension of E_0, in $\mathbf{R}$ and E_1 in $\mathbf{R}^2$.

Hint: Find the distance d*, between two adjacent elements and then cover with balls of radius* d*, then use the definition.*

F-1 Let K be a compact metric space and denote by $X := C(K)$ the Banach space of continuous, complex valued functions on K (equipped with the maximum norm) and fix $a \in X$. Define the operator $A : X \to X$ by

$$(Af)(t) := a(t)f(t) \text{ for } t \in K.$$

(a) Determine $\sigma(A)$.

(b) Give a sufficient condition on a such that $\sigma_p(A) \neq \emptyset$.

(c) Give an example of K and a for which $\sigma_p(A) = \emptyset$.

F-2 Let $\mathcal{H}$ be a Hilbert space and $B \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator.

(a) Show that the set $\sigma(B)$ is compact.

(b) Show that $B = B^*$ is a projection if and only if $\sigma(B) \subset \{0, 1\}$.

Hint: Theorems from the lecture can be used without proof, but it need to clear how they are used!

Please turn!

F-3 Let $\mathcal{H}$ be a Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *normal* if $TT^* = T^*T$.

(a) Show: A linear operator T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in \mathcal{H}$.

(b) Show that for normal operators the following hold:

i. $\ker(T) = \ker(T^*)$

ii. $\mathrm{ran}(T)$ is dense if and only if T is injective.

iii. If $Tx = \mu x$ for some $x \in \mathcal{H}$ and $\mu \in \mathbb{C}$, then $T^*x = \overline{\mu}x$.

iv. If μ and λ are distinct eigenvalues of T, then the corresponding eigenvectors are orthogonal.

After correction the marked exams can be picked up in studentexpeditionen, house 6 (SU).

Good luck!

Solution to problem I-1) Define $f(x) = \chi_E(x)$, and $g(x) = \int_0^x f(y)dy$. Then by Lebesgue differentiation theorem we have $g' = f$ a.e. Now by definition, for each z

$$g'(z) \approx \frac{1}{r}\int_z^{z+r} f(y)dy = \frac{1}{r}\int_z^{z+r} \chi_E(y)dy = \frac{1}{r}\mu(I_r(z)\cap E) =$$

$$\frac{1}{r}(r - \mu(I_r(z)\setminus E)) = 1 - \frac{1}{r}\mu(I_r(z)\setminus E) < \frac{99}{100}.$$

As r tends to zero we obtain $g'(z) < 1$ for all $z \in E$. This violates Lebesgue differentiation theorem, unless $\mu(E) = 0$.

Solution to problem I-2) See the book

Solution to problem I-3a)

Falconer-Fractal Geometry. Mathematical Foundations and Applications- Page 33 of file. and also page 36-37 of the file.

The distance between two points in the set E_0 is approximately j^{-m-1}. Hence for $\epsilon > 0$ we can cover the set E_0 by approximately $(1/\epsilon)^{1/m+1}$ balls of radius ϵ. So by definition of Minkowski dimension (or Box dimension) we need

$$\lim_{\epsilon\to 0} N(E_0,\epsilon)\epsilon^s = \lim_{\epsilon\to 0}(1/\epsilon)^{1/m+1}\epsilon^s = 0$$

which is possible only if $s > 1/(m+1)$. Hence the infimum of all such s is indeed $1/(m+1)$.

For more reading see: *Falconer-Fractal Geometry. Mathematical Foundations and Applications, Page 33 of file. and also page 36-37 of the file.*

Solution to problem I-3b) Similar analysis for this case just adds one more dimension, and we have $(m+2)/(m+1)$. You have to think that in the direction of y-axis you have to take $1/\epsilon$ number of balls.

Remark)

Also note that as m gets larger the dimension decreases. Hence for points tending to the origin exponentially, we must have zero dimension.

F1 $X := C(K)$ K.. compact metric space

$a \in X$

$(Af)(t) := a(t) f(t) \quad t \in K$

a) $\lambda \in \mathbb{C}$: $(A-\lambda) f = g$

$$(a(t) - \lambda) f(t) = g(t)$$

1) $\lambda \notin \operatorname{ran} a \Rightarrow a(t) - \lambda \neq 0 \quad \forall t \in K$

$$\Rightarrow f(t) = \frac{g(t)}{a(t) - \lambda} \in C(K)$$

$\Rightarrow$ $A - \lambda$ is boundedly invertible ; $\lambda \in \varrho(A)$

2) $\lambda \notin \operatorname{ran} a \Rightarrow \exists\, t_0 \in K : a(t_0) = \lambda$

$\Rightarrow g(t_0) = 0$ i.e. $(A-\lambda) f = g$

is not solvable for all $g \in C(K)$

$\Rightarrow \lambda \notin \varrho(A)$

Answer: $\sigma(A) = \operatorname{ran} a \;(= \{ a(t) : t \in K \})$

b) $\lambda \in \sigma_p(A) \Leftrightarrow$ $(a(t) - \lambda) f(t) = 0$ has a non-trivial solution

Sufficient condition for $\lambda \in \sigma_p(A)$: $\exists$ ball $B_0 \subset K$ such that

$a|_{B_0} \equiv \lambda$ (locally constant)

Hence, $\exists\, B_1 \subseteq B_0$ and $f \in C(K)$ such that $f(t) = \begin{cases} 0 & t \notin B_0 \\ 1 & t \in B_1 \end{cases}$

i.e. f is eigenvector

c) $K = [0,1]$, $a(t) = t \Rightarrow \sigma_p(A) = \emptyset$

F-2 $B \in \mathcal{B}(\mathcal{H})$

(a) Show: $\sigma(B)$ is compact

(1) $\varrho(B)$ is open (Thm. from the lecture)

$\Rightarrow \sigma(B) = \mathbb{C} \setminus \varrho(B)$ closed

(2) $\lambda \in \mathbb{C}$ with $|\lambda| > \|B\|$

$\Rightarrow (B - \lambda) = \lambda \left(\frac{1}{\lambda} B - I \right)$ with $\| \frac{1}{\lambda} B \| < 1$

is boundedly invertible (von Neuman-series Theorem from ARA 1)

$\Rightarrow \sigma(B)$ bounded

$\Rightarrow \sigma(B)$ compact

(b) $B = B^*$. Show $B = B^2 \Leftrightarrow \sigma(B) \subseteq \{0, 1\}$

$\Rightarrow$ B projection $\Rightarrow$ if $x \in \mathcal{H}$, then $x = x_{\parallel} + x_{\perp}$ where

$x_{\parallel} \in \operatorname{ran} B$ (i.e. $Bx_{\parallel} = x_{\parallel}$)
$x_{\perp} \in \ker B$

$(B - \lambda) x = y$

$(B - \lambda)(x_{\parallel} + x_{\perp}) = y_{\parallel} + y_{\perp}$

$(1 - \lambda) x_{\parallel} - \lambda x_{\perp} = y_{\parallel} + y_{\perp} \Leftrightarrow (1 - \lambda) x_{\parallel} = y_{\parallel}$ and $-\lambda x_{\perp} = y_{\perp}$

Hence: $\lambda \notin \{0, 1\} \Rightarrow \lambda \in \varrho(A)$

i.e. $\sigma(B) \subseteq \{0, 1\}$

"$\Leftarrow$" $B = B^*$ and $\sigma(B) \subseteq \{0,1\}$

$\underset{\substack{\text{Spectral}\\\text{theorem}}}{\Longrightarrow}$ $B = \int_{\mathbb{R}} t \, dE_t = 0 \cdot E_0 + 1 \cdot E_1$

orthogonal projections $\square$

F-3 $TT^* = T^*T$ normal

(a) Show: T normal $\Leftrightarrow$ $\|Tx\| = \|T^*x\|$ $\forall x \in \mathcal{H}$

"$\Rightarrow$" $\|Tx\|^2 = (Tx,Tx) = (T^*Tx,x) \underset{\text{normal}}{=} (TT^*x,x) = (T^*x,T^*x) = \|T^*x\|^2$

"$\Leftarrow$" $(T^*Tx,x) = (TT^*x,x)$

Show more generally: $A,B \in \mathcal{B}(\mathcal{H})$ $\quad A = A^*, B = B^*$

$(Ax,x) = (Bx,x)$ $\forall x \in \mathcal{H} \Rightarrow A = B$

- $(A(x+y),x+y) = (B(x+y),x+y)$

$\Rightarrow (Ax,y) + (Ay,x) = (Bx,y) + (By,x)$

$\Rightarrow \operatorname{Re}(Ax,y) = \operatorname{Re}(Bx,y)$

- $(A(x+iy),x+iy) = (B(x+iy),x+iy)$

$\Rightarrow -(Ax,y) + (Ay,x) = -(Bx,y) + (By,x)$

$\Rightarrow \operatorname{Im}(Ax,y) = \operatorname{Im}(Bx,y)$

$(Ax,y) = (Bx,y)$ $\quad \forall y, \forall x$

$\Rightarrow Ax = Bx$ $\forall x$ $\Rightarrow$ $A = B$

(b) $\ker T = \ker T^*$ (follows directly from (a))

(c) $\operatorname{ran} T$ dense $\iff T$ inj.

$\operatorname{ran} T$ dense $\iff (\operatorname{ran} T)^\perp = \{0\} \underset{\substack{(\operatorname{ran}T)^\perp = \\ \ker T^*}}{\iff} \ker T^* = \{0\} \underset{(b)}{\iff} \ker T = \{0\}$

(d) $Tx = \mu x$ i.e. $x \in \ker(T - \mu) \underset{(b)}{=} \ker(T^* - \bar{\mu})$

i.e. $T^* x = T^* \bar{\mu}$

(e) $Tx = \lambda x$, $Ty = \mu y$

$\lambda (x, y) = (Tx, y) = (x, T^* y) \underset{(d)}{=} (x, \bar{\mu} y) = \mu (x, y)$

$\underset{\lambda \neq \mu}{\Rightarrow} (x, y) = 0$ i.e. $x \perp \mu$