MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Annemarie Luger (SU),
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Written exam in
Advanced Real Analysis II,
MM8039 (SF2744)
August 20, 2015
9:00-14:00

The exam consists of two parts:

i) Written exam, which consists of 6 problems (4 points each) and gives you grade C at maximum.
ii) Oral exam, which consists of 2 questions (3 points each). A minimum grade Fx is required for eligibility of the oral exam.

Bonus: *Already passed homework and midterm exam can replace questions according to:*
Passed Homework 1 replaces question 1 in the written exam.
Passed Midterm replaces question 2 in the written exam.
Passed Homework 2 replaces question 3 in the written in exam.

Credit scale:

A= at least 27,5 points,	*B= at least 23,5 points,*	*C= at least 21 points,*
D= at least 18 points,	*E= at least 15 points*	*Fx= at least 13,5 points.*

Important: *If you intend to take the oral exam, email to luger@math.su.se no later than Monday 24/8.*

Motivate your solutions carefully!!!

1. Let X and Y be Banach spaces and $T \in \mathcal{B}(X,Y)$ and denote (as in the lecture notes) the dual operator by $T^+ : Y^* \to X^*$. (Note: in [Friedman] for this operator the notation T^* was used).
   (a) Let $X = Y = \ell^p(\mathbb{N})$ for $1 \le p < \infty$ such that $X^* = \ell^q(\mathbb{N})$ with $\frac{1}{p} + \frac{1}{q} = 1$. Consider the right shift $S \in \mathcal{B}(\ell^p(\mathbb{N}))$ given by
   $$Sx := (0, x_1, x_2, \ldots).$$
   Determine the dual operator S^+.
   (b) If $X = Y$ is a Hilbert space there is also the notion of the adjoint operator $T^* \in \mathcal{B}(X)$. Explain the relation between T^* and T^+ in this case. State all definitions that you are using!

2. Let $\mathcal{H}$ be a Hilbert space and $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator, i.e. $U^* = U^{-1}$.
   (a) Show that all eigenvalues of U lie on the unit circle.
   (b) Show that for $\lambda_0 \in \mathbb{C} \setminus \{0\}$ it holds
   $$\lambda_0 \in \sigma_r(U) \qquad \Longrightarrow \qquad \frac{1}{\overline{\lambda_0}} \in \sigma_p(U).$$
   (c) Which conclusion can be drawn from (a) and (b)?

3. Prove the following standard properties for Radon-Nykodym derivatives:
   (a) Linearity: $\frac{d(c_1\nu_1 + c_2\nu_2)}{d\mu} = c_1 \frac{d\nu_1}{d\mu} + c_2 \frac{d\nu_2}{d\mu}$, $c_1, c_2 \in \mathbb{R}$.
   (b) Change of measure: If $\nu \ll \mu$ and g is a ν-integrable function then $\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$.
   (c) Chain Rule: If $\lambda \ll \nu \ll \mu$ then $\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu} \frac{d\nu}{d\mu}$.

Please turn!

4. Let m be the Lebesgue measure on the interval $[0, +\infty)$, and let μ be a finite (unsigned) measure. Recall the definition of a signed measure μ to be continuous if $\mu(\{x\}) = 0$ for all singletons x. Let also $k(x)$ be an increasing continuous function, with $k(0) = 0$.

   (a) Show that μ is a continuous measure if and only if the function $g(x) := \mu([0, k(x)])$ is continuous.

   (b) Show that μ is an absolutely continuous measure with respect to m if and only if the function $g(x) = \mu([0, k(x)])$ is absolutely continuous.

5. Decide the Hausdorff and Minkowski dimension of the graph of the following functions, for $x > 0$:

$$\textbf{a)}\ y = \log(1+x)\sin\frac{1}{x}, \qquad \textbf{b)}\ y = \frac{1}{x}\sin\frac{1}{x}.$$

   Motivate your answers carefully with proofs.

6. (a) Let a function $K \in C\big([a,b] \times [a,b]\big)$ be given and define

$$\big(Bf\big)(x) := \int_a^b K(x,y) f(y)\, dy.$$

   Show that B is a compact operator in $L^2[a,b]$, i.e. $B \in \mathcal{K}\big(L^2[a,b]\big)$.

   (b) Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{K}(\mathcal{H})$ be a self adjoint, compact operator. Describe properties of the spectrum of A, i.e what kind of spectrum and where it lies.

   (c) Explain how the results from the previous subquestion can be used in the theory of Sturm-Liouville operators, in particular, be precise about the compact operator in this connection. (You do not have to give all proofs but some are needed in order to get full points!)

If you want to know your result write an email to luger@math.su.se. After correction the marked exams can be picked up in studentexpeditionen, house 6, room 204 (SU).

Good luck!