STOCKHOLM UNIVERSITY
Department of Mathematics
MM8039 (SF2744)
Examinators: Salvador Rodríguez-López
Henrik Shahgholian

Resit Written Exam
Advanced Real Analysis II
19 August, 2019
09:00-14:00

Credit scale: A maximum of 30 points can be obtained from the written exam including bonus from Homework: The written exam give a maximum of 24 points $(3+3)\times 4$. The homework gives a maximum of 6 points $(3+3)$. Each question has a maximum of 4 points, and grading is set according to:

A= at least 26,5 points	C= at least 20 points	E= at least 15 points
B= at least 23 points	D= at least 17,5 points	Fx= at least 13,5 points.

I-1. Let μ be the Lebesgue-Stieltjes measure associated with $f(x) = x + \chi_{\{0\}}(x)$, i.e.

$$\mu(E) = \int_E dF, \qquad \text{for all Lebesgue measurable sets } E \subset \mathbb{R}.$$

Find the Lebesgue decomposition of μ with respect to Lebesgue measure on $\mathbb{R}$.

I-2. State and prove the Radon-Nikodym theorem.

I-3. Let (X, μ) be a σ-finite measure space and $1 < p < \infty$, with $1/p + 1/q = 1$. Prove that the conjugate of $L^p(X, \mu)$ is $L^q(X, \mu)$.

F-1. (a) Let X, Y be two Banach spaces. Given $T \in \mathscr{B}(X, Y)$, how does the adjoint $T^\times$ is defined? Prove that $\|T^\times\| = \|T\|$?

(b) Consider the operator T defined for $f \in L^{4/3}([0,1])$ by

$$Tf(x) = \int_0^1 e^{ixy} f(y) \mathrm{d}y.$$

Show that $T \in \mathscr{B}(L^{4/3}([0,1]))$ and calculate its adjoint.

F-2. Let X be a Banach space, and let $T \in \mathscr{B}(X)$.

(a) Give the definition of the set of regular points $\rho(T)$, of T. For all $\mu \in \rho(T)$, give the definition of the resolvent operator $R(\mu)$.

(b) Show that $\rho(T)$ is an open set in $\mathbb{C}$ and prove that for all $\lambda, \mu \in \rho(T)$ one has

$$R(\mu)R(\nu) = R(\nu)R(\mu).$$

F-3. Let

$$K(x,y) = \begin{cases} y(1-x) & \text{if } 0 \le y \le x \le 1, \\ x(1-y) & \text{if } 0 \le x < y \le 1, \end{cases}$$

and $\tilde{f}(x) = \int_0^1 K(x,y) f(y) \mathrm{d}y$. Define

$$T_1 : \mathscr{C}[0,1] \to \mathscr{C}[0,1], \qquad T_2 : L^2[0,1] \to L^2[0,1]$$

by $T_i f = \tilde{f}$, for $i = 1, 2$.

(a) Are T_1 and/or T_2 compact? Is $\sigma(T_1) = \sigma(T_2)$? Detailed motivation is needed.

(b) Determine the non-zero eigenvalues of T_2 and their respective multiplicity. Give the spectral representation of T_2.

GOOD LUCK