STOCKHOLM UNIVERSITY
Department of Mathematics
MM8039 (SF2744)
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Resit Written Exam
Advanced Real Analysis II
19 August, 2019
09:00-14:00

Credit scale: A maximum of 30 points can be obtained from the written exam including bonus from Homework: The written exam give a maximum of 24 points $(3+3)\times 4$. The homework gives a maximum of 6 points $(3+3)$. Each question has a maximum of 4 points, and grading is set according to:

A= at least 26,5 points	C= at least 20 points	E= at least 15 points
B= at least 23 points	D= at least 17,5 points	Fx= at least 13,5 points.

I-1. Let μ be the Lebesgue-Stieltjes measure associated with $f(x) = x + \chi_{\{0\}}(x)$, i.e.

$$\mu(E) = \int_E dF, \qquad \text{for all Lebesgue measurable sets } E \subset \mathbb{R}.$$

Find the Lebesgue decomposition of μ with respect to Lebesgue measure on $\mathbb{R}$.

I-2. State and prove the Radon-Nikodym theorem.

I-3. Let (X, μ) be a σ-finite measure space and $1 < p < \infty$, with $1/p + 1/q = 1$. Prove that the conjugate of $L^p(X, \mu)$ is $L^q(X, \mu)$.

F-1. (a) Let X, Y be two Banach spaces. Given $T \in \mathscr{B}(X, Y)$, how does the adjoint $T^\times$ is defined? Prove that $\|T^\times\| = \|T\|$?

(b) Consider the operator T defined for $f \in L^{4/3}([0, 1])$ by

$$Tf(x) = \int_0^1 e^{ixy} f(y)\mathrm{d}y.$$

Show that $T \in \mathscr{B}(L^{4/3}([0, 1]))$ and calculate its adjoint.

F-2. Let X be a Banach space, and let $T \in \mathscr{B}(X)$.

(a) Give the definition of the set of regular points $\rho(T)$, of T. For all $\mu \in \rho(T)$, give the definition of the resolvent operator $R(\mu)$.

(b) Show that $\rho(T)$ is an open set in $\mathbb{C}$ and prove that for all $\lambda, \mu \in \rho(T)$ one has

$$R(\mu)R(\nu) = R(\nu)R(\mu).$$

F-3. Let

$$K(x, y) = \begin{cases} y(1-x) & \text{if } 0 \le y \le x \le 1, \\ x(1-y) & \text{if } 0 \le x < y \le 1, \end{cases}$$

and $\tilde{f}(x) = \int_0^1 K(x, y)f(y)\mathrm{d}y$. Define

$$T_1 : \mathscr{C}[0, 1] \to \mathscr{C}[0, 1], \qquad T_2 : L^2[0, 1] \to L^2[0, 1]$$

by $T_i f = \tilde{f}$, for $i = 1, 2$.

(a) Are T_1 and/or T_2 compact? Is $\sigma(T_1) = \sigma(T_2)$? Detailed motivation is needed.

(b) Determine the non-zero eigenvalues of T_2 and their respective multiplicity. Give the spectral representation of T_2.

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Suggestions for solutions

Solution to problem 1) The Lebsgue decomposition of μ is $\delta_0 + dx$. To justify this one needs to apply μ to any open interval (a, b) to arrive at $\mu(a, b) = b - a + 1$ if $0 \in (a, b)$, and otherwise $\mu(a, b) = b - a$. All to all one obtains $(\delta_0 + dx)(a, b) = F(b) - F(a)$ and similarly $\mu(a, b) = F(b) - F(a)$, and the uniqueness gives the representation $\mu = \delta_0 + dx$.

Solution to problem 2, 3) See the text book.

Solution to problem 4)

1. See the lecture notes of the course.
2. Clearly $Tf(x) \in L^\infty[0, 1] \subset L^{4/3}[0, 1]$.

   We know we can identify $(L^{4/3})^*$ with L^4, in the sense that for all $\Lambda \in (L^{4/3})^*$, there exists $g_\Lambda \in L^4$ such that
   $$\Lambda(f) := \int_0^1 f(y) g_\Lambda(y) \mathrm{d}y.$$
   In this way, for all $g \in L^4 = (L^{4/3})^*$ and for all $f \in L^{4/3}$
   $$(T^\times g)(f) := g(Tf) = \int_0^1 Tf(y) g(y) \mathrm{d}y = \int_0^1 f(x) \int_0^1 e^{iyx} g(y) \mathrm{d}y \mathrm{d}x.$$
   So we have that
   $$T^\times g(x) = \int_0^1 e^{iyx} g(y) \mathrm{d}y = Tg(x).$$

Solution to problem 5)

1. See the lecture notes of the course.
2. See the lecture notes of the course.

Solution to problem 6)

1. Observe that
   $$K(x, y) \in \mathscr{C}([0, 1]^2, \mathbb{R}) \subset L^2([0, 1]^2).$$
   and also that it satisfies
   $$K(x, y) = K(y, x) = \overline{K(y, x)}.$$
   So it follows that T_1 is a Fredholm operator, and hence compact, as well as that T_2 is a self-adjoint Hilbert-Schmidt operator, and hence its is compact.

   In particular we know that
   $$\sigma(T_j) \setminus \{0\} = \sigma_p(T_j) \setminus \{0\}$$
   for $j = 1, 2$, and that $\{0\}$ is the only possible element if the continuous spectrum of these operators.

   Moreover, since these operators are compact between infinite dimensional spaces, they can't be invertible, so $0 \in \sigma(T_j)$.

   We can explicitly write
   $$\tilde{f}(x) = (1 - x) \int_0^x y f(y) \mathrm{d}y + x \int_x^1 (1 - y) f(y) \mathrm{d}y.$$
   Hence, if $f \in L^2(\mathbb{R})$, it follows by the DCT, that $\tilde{f}$ is continuous. In particular, by a bootstrap argument, any eigenvector of non-zero eigenvalue, of either T_1 or T_2, is a smooth function. So, it follows that
   $$\sigma_p(T_1) \setminus \{0\} = \sigma_p(T_2) \setminus \{0\}.$$

We can also observe that if f is an igenvector of eigenvalue $\{0\}$ of either operator, differentiating twice we see that f must satisfy that

$$0 = -\int_0^x yf(y)\mathrm{d}y + \int_x^1 f(y)(1-y)\mathrm{d}y,$$

and that

$$0 = -xf(x) + (x-1)f(x) = -f(x),$$

for a.e. $x \in [0,1]$. This implies that $f \equiv 0$. In other words $0 \notin \sigma_p(T_j)$.

As a consequence, we have that

$$\sigma(T_1) = \sigma(T_2).$$

2. Now, if $\lambda \in \sigma_p(T_1) \setminus \{0\}$, we have that f must satisfy the second order ODE

$$\lambda f''(x) = -f(x),$$

with boundary conditions $f(1) = 0$ and $f(0) = 0$. The non-trivial solutions of this BVP are given by constant multiples of

$$f_k(x) = \sin(k\pi x), \qquad k = 1, 2, 3\ldots,$$

where $\lambda_k = (k\pi)^{-2}$, which are eigenvalues of multiplicity one.

A calculation shows

$$\int_0^1 f_k(x)^2 \mathrm{d}x = \frac{1}{2},$$

and we know that two eigenvectors of different eigenvalues are orthogonal to each other. Hence

$$g_k(x) = \sqrt{2}\sin(k\pi x),$$

form an orthonormal system in $L^2[0,1]$.

By the Hilbert-Schmidt theorem, since we have shown that $\ker T_2 = \{0\}$, we have that $\{g_k\}_{k\geq 1}$ is a orthonormal basis of $L^2[0,1]$. And we also have that for all $f \in L^2[0,1]$

$$T_2 f(x) = \sum_{k\geq 1} \frac{2}{k^2\pi^2} \langle f, f_k\rangle \sin(k\pi x),$$

where

$$\langle f, f_k\rangle = \int_0^1 f(x)\sin(k\pi x)\mathrm{d}x.$$