MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: A.A. Sola

Final examination in
Mathematics III Foundations of Analysis
7.5 hp
February 17th 2021

You are not permitted to collaborate with other students or consult other individuals. Maximum total score is 20 points: 15 points and participation in the oral examination are required to pass. See course webpage for full details.
Appropriate amounts of detail are required for full marks.

1. Determine which of the following statements are true, and which are false. Explain your reasoning.
   (a) If f is a differentiable real-valued function on the interval $[a, b]$ whose derivative is non-zero in (a, b) then f does not attain a largest value on $[a, b]$.
   (b) If a sequence of real-valued functions $\{f_n\}$ converges uniformly on $\mathbb{R}$ to a differentiable function then there exists an integer N such that, for each $n \geq N$, the function f_n is differentiable.
   (c) If a set K is compact subset of the real line, then either K contains some closed interval or else K is a countable union of points.
   (d) If the real power series $\sum_{k=1}^{\infty} a_k x^k$ converges at some point $x_0 \in \mathbb{R}$ then there exists some neighborhood containing x_0 on which the series converges.
   (e) The set of real-valued differentiable functions on $[0, 1]$ equipped with the function
   $$d(f, g) = \int_0^1 |f'(x) - g'(x)|^2 dx$$
   is an example of a complete metric space.

2. A subset $E \subset \mathbb{R}^2$ is said to be *complement-connected* if its complement E^c is connected.
   (a) Give an example of a set in $\mathbb{R}^2$ that is connected but not complement-connected.
   (b) If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a bounded continuous function and E is a complement-connected set, need $f(E)$ be complement-connected?

3. Compute the Riemann-Stieltjes integral
   $$\int_0^{\frac{\pi}{2}} f d\alpha$$
   where $f(x) = x^2$ and
   $$\alpha(x) = \begin{cases} \sin x, & 0 \leq x \leq \frac{\pi}{4} \\ 1 + \sin x, & \frac{\pi}{4} < x \leq \frac{\pi}{2} \end{cases}.$$

4. Let $f : [-\pi, \pi]$ be a continuously differentiable function. Define a sequence $\{\hat{f}(n)\}_{n \in \mathbb{Z}}$ via
   $$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$
   What, if anything, can you say about
   - (a) the boundedness of the sequence $\{\hat{f}(n)\}$?
   - (b) the relation between the sequences $\{\hat{f}(n)\}$ and $\{\widehat{f'}(n)\}$?
   - (c) the limiting behavior of $\{\hat{f}(n)\}$ as $n \to \infty$?