Logic Exam Solution.

1) This is a theory question & we refer to the literature

2) a)

$V(P_1)=1$
$V(P_3)=1$
$V(P_2)=0$

b)

$V_2(P_1)=0$	$V_1(P_1)=0$
$V_2(P_2)=0$	$V_1(P_2)=1$
$V_2(P_3)=0$	$V_2(P_3)=1$

3) The possible term of the language are

(a) Variables $\cup \{f_1, f_2\}$

They are all free for x_i in φ since φ does not inulue any quantifier

(b) If t is a variable $t=x_i$

$\mathcal{A}=\langle \{a,b,c\}; P=\{a\}; f_1=b, f_2=c \rangle$

$v: N \longrightarrow A$
$n \longmapsto a$

Gives a countermodel

if $t=f_i$

$\mathcal{A}=\langle \{a\,b\,c\}; P=\{a\}, f_i=a, f_j=b \}$
$j \neq i$

Gives a countermodel.

4) a) This is a theory question & we refer to the book.

b) Given the definitions we need that

① s is not free for x_i in $R(x_0)$

$\boxed{\text{OR}}$

② all the following must hold

a) $i \neq 0$

b) $x_i \in FV(\varphi)$

c) $x_0 \in V(t)$

Since all terms t are free for x_i in $R(x_0)$ we have to satisfy 2. But this is impossible: since $FV(\varphi) = \emptyset$ and $x_i \in FV(\varphi)$ cannot be true

CONCLUSION : there is no such term.

c) We will show that for every formula φ x_i is free for x_i in φ

→ True for atomic formulas

→ If it is true for some formula φ_1 φ_2 then it is true for

$\varphi_1 \to \varphi_2$

$\varphi_1 \vee \varphi_2$

$\varphi_1 \wedge \varphi_2$

→ Suppose it true for σ then we will show it for $\exists x_j \sigma$ and $\forall x_j \sigma$

We need one of the following

1) $x_i = x_j$
2) $x_i \notin FV(\varphi)$
3) $x_j \notin V(t)$

Suppose that 3) does not hold, then
$x_j = x_i$. so 1) holds

5) (a)

$$\dfrac{\dfrac{\dfrac{\neg \exists x_0 \varphi \qquad \dfrac{[\varphi]^1}{\exists x_0 \varphi}\ \exists I}{\bot}\ \to E}{\neg \varphi}\ \to I_1}{\forall x_0\, \neg \varphi}\ \forall I$$

(observe x_0 is not free in any of the UA)

(b)

$$\dfrac{\dfrac{\dfrac{\neg (P_1 \wedge P_2) \qquad \dfrac{[P_2]^2 \qquad [P_1]^1}{P_1 \wedge P_2}\ \wedge I}{\bot}\ \to E}{\neg P_1}\ \to I_1}{P_2 \to \neg P_1}\ \to I_2$$

c) There is a countermodel

$V(P_1)=0$
$V(P_2)=1$ $\Rightarrow [\![P_1 \vee \neg P_2]\!]^{v} = 0$

but

$[\![\neg P_1 \to P_2]\!] = 1 \to 1 = 1$

6) $\exists x_1 \varphi$ is closed so we have to show that it is true in any structure

$\mathcal{A} = \langle A, P, f_1, f_2 \rangle$

independently from the interpretation of variables

There are two cases

① $P = A$ then we have that

$[\![P(f_1)]\!]^{\mathcal{A}} = [\![P(f_2)]\!]^{\mathcal{A}} = 1$

no matter the value of f_1 and f_2.

thus $[\![P(x_1) \to (P(f_1) \wedge P(f_2))]\!]^{\mathcal{A} v(x_1 \to a)} = 1$

for all $a \in A$ and in particular

$[\![\exists x_1 P(x_1) \to (P(f_1) \wedge P(f_2))]\!]^{\mathcal{A} v} = 1$

② $P \neq A$ then there is $a \in A$ such that

$[\![P(x_1)]\!]^{\mathcal{A} v(x_1 \to a)} = 0$

we conclude that

$[\![P(x_1) \to (P(f_1) \wedge P(f_2))]\!]^{\mathcal{A} v(x_1 \to a)} = 1$

for some $a \in A$

Thus

$[\![\exists x_1 (P(x_1) \to (P(f_1) \wedge P(f_2))]\!]^{\mathcal{A} v} = 1.$

b) Since $\exists x_1 \varphi$ is a tautology we have

$\emptyset \models \exists x_1 \varphi$

by completeness

$\emptyset \vdash \exists x_1 \varphi$

in order to have the existence property we need that

$\emptyset \vdash \varphi[t/x_1]$ for some t free for x_1 in φ

but, by soundness this would imply

$\emptyset \models \varphi[t/x_i]$ and so that

$\varphi[t/x_i]$ is a tautology,

which is impossible since φ has a countermodel

7) a) $V(\varphi) = \{x_1, x_2\}$

$FV(\varphi) = \{x_2\}$

b) The formula is closed so its evaluation does not depend on the evaluation

$$[\![\forall x_2 \forall x_1 (P(x_1 x_2) \rightarrow \forall x_1 P(x_1 x_2))]\!]^{A\,v} = 1$$

$$\Leftrightarrow 1 = [\![P(x_1, x_2)]\!]^{A\,v(x_1 \rightarrow x)(x_2 \rightarrow y)} \longrightarrow [\![\forall x_1 P(x_1, x_2)]\!]^{A\,v(x_1 \rightarrow x)(x_2 \rightarrow y)}$$

for all x, y in A

If $y = b$ then $[\![P(x\, x_2)]\!]^{A\,v(x_1 \rightarrow x)(x_2 \rightarrow b)} = 0$

& so the implication is true

if $y \neq b$ we have that

$[\![P(x_1 x_2)]\!]^{A\,v(x_1 \rightarrow x)(x_2 \rightarrow y)} = 1$ for all x

in particular

$$[\![\forall x_1\, P(x_1,x_2)]\!]^{A,v(x_2\to y)} = 1$$

and we have that

$$[\![\forall x_2\, \varphi]\!]^{Av} = 1$$

(c) $P' = \{(a\,c)\}$

$$[\![\forall x_1\, P(x_1,x_2)]\!]^{Av} = 1$$

iff $[\![P(x_1,x_2)]\!]^{Av(x_1\to x)} = 1$ for all x

but $(b\,c) \notin P$

So $[\![P(x_1,x_2)]\!]^{Av(x_1\to b)} = 0$

~~iff~~

$$[\![P(x_1,x_2) \to \forall x_1\, P(x_1,x_2)]\!]^{Av(x_1\to a)}$$

$$\shortparallel$$

$$\underbrace{[\![P(x_1,x_2)]\!]^{Av(x_1\to a)}}_{=\,1} \to \underbrace{[\![\forall x_1\, P(x_1,x_2)]\!]^{Av(x_1\to a)}}_{=\,0}$$

$$= 0$$

8) $\exists x_1 \exists x_2\ x_1 \doteq x_2$ is a tautology so we can derive it without any assumptions.

$$\dfrac{[x_1 \doteq x_2]^1}{\exists x_2\ x_1 \doteq x_2}\ \exists I$$

$$\dfrac{\exists x_2\ x_1 \doteq x_2}{\exists x_1 \exists x_2\ x_1 \doteq x_2}\ \exists I$$

$$\dfrac{[\neg \exists x_1 \exists x_2\ x_1 \doteq x_2]^3 \qquad \exists x_1 \exists x_2\ x_1 \doteq x_2}{\bot}\ \to E$$

$$\dfrac{\bot}{\neg (x_1 \doteq x_2)}\ \to I_1$$

$$\dfrac{\neg (x_1 \doteq x_2)}{\forall x_3\ \neg x_3 \doteq x_2}\ \forall I$$

$$\dfrac{\forall x_3\ \neg x_3 \doteq x_2}{\neg x_3 \doteq x_2}\ \forall E$$

$$\dfrac{\neg x_3 \doteq x_2 \qquad [x_3 \doteq x_2]^2}{\bot}\ \to E$$

$$\dfrac{}{x_2 \doteq x_2}\ REF$$

$$\dfrac{x_2 \doteq x_2}{\exists x_3\ x_3 \doteq x_2}\ \exists I$$

$$\dfrac{\exists x_3\ x_3 \doteq x_2 \qquad \bot}{\bot}\ \exists E_2$$

$$\dfrac{\bot}{\exists x_1 \exists x_2\ x_1 \doteq x_2}\ RAA_3$$

9) (a) both $P_1 \to P_2 \vdash P_3$
and $P_1 \to P_2 \vdash \neg P_3$
have countermodels

(b) Theory question