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Suggested solutions

Exam: Introduction to Finance Mathematics (MT5009), 2023-05-17

Problem 1

(A) The value is

$$\frac{20}{1.03^1} + \frac{20}{1.03^2} + \frac{120}{1.03^3} = 148,0864.$$

(B) We look for a constant C such that

$$C \sum_{i=1}^{5} (1+R)^i = 400000,$$

which is equivalent to

$$C = \frac{400000}{\sum_{i=0}^{5} (1+R)^i - 1}.$$

Since

$$\sum_{i=0}^{n} (1+R)^i = \frac{(1+R)^{n+1} - 1}{R}$$

we thus obtain

$$C = \frac{400.000}{\frac{(1+0.03)^6 - 1}{0.03} - 1} \approx 73147.406.$$

Problem 2

(A) Using the notation in Capinski & Zastawniak, we have

$$C = \begin{pmatrix} \sigma_1^2 & c_{12} \\ c_{12} & \sigma_2^2 \end{pmatrix},$$

so that

$$C^{-1} = \frac{1}{\sigma_1^2 \sigma_2^2 - c_{12}^2} \begin{pmatrix} \sigma_2^2 & -c_{12} \\ -c_{12} & \sigma_1^2 \end{pmatrix}.$$

Using the above we find with basic calculations

$$w_{MVP} = \frac{uC^{-1}}{uC^{-1}u^T} = \frac{1}{\sigma_1^2 + \sigma_2^2 - 2c_{12}} (\sigma_2^2 - c_{12} \quad \sigma_1^2 - c_{12}).$$

Using $c_{12} = p_{12}\sigma_1\sigma_2$ we obtain by plugging in the numbers

$$w_{MVP} \approx (0.7660 \quad 0.2340)$$

(B) The optimization problem is:

$$\begin{array}{cc} \text{minimize}_w & \text{wCw}^T \\ \text{subject to} & wu^T = 1. \end{array}$$

The constraint is equivalent to setting $w = (1-x, x)$ for some x (to see this use that $u = (1, 1)$ by definition). Hence, the problem is equivalent to maximizing

$$wCw^T = (1-x)^2\sigma_1^2 + 2c_{12}x(1-x) + \sigma_2^2x^2$$

over x. Differentiating this expression and setting it to zero (which clearly gives the minimum) gives

$$x(2\sigma_1^2 + 2\sigma_2^2 - 4c_{12}) - 2\sigma_1^2 + 2c_{12} = 0.$$

This gives us

$$x = \frac{\sigma_1^2 - c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}$$

which in turn gives (with basic calculations)

$$1 - x = \frac{\sigma_2^2 - c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}$$

so that the solution to the problem is

$$\begin{aligned} w_{mvp} &= (1-x, x) \\ &= \frac{1}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}(\sigma_2^2 - c_{12}, \sigma_1^2 - c_{12}). \end{aligned}$$

(Note that this is in line with the answer in **(A)**).

Problem 3

(A) The replicating portfolio satisfies by definition

$$\begin{aligned} x(1)S^u + y(1)(1+R) &= max\{S^u - X; 0\} \\ x(1)S^d + y(1)(1+R) &= max\{S^d - X; 0\}, \end{aligned}$$

where $x(1)$ is the number of shares and $y(1)$ is the amount of money in the risk-free asset in the replicating portfolio.

Note that $S^u = S(0)(1+U) = 12$ and $S^u = S(0)(1+D) = 9$. By plugging in numbers we see that the replicating portfolio should satisfy

$$\begin{aligned} x(1)12 + y(1)1.1 &= 2 \\ x(1)9 + y(1)1.1 &= 0. \end{aligned}$$

Solving this equation system gives the replicating portfolio $x(1) = 0.6667$ and $y(1) = -5.4545$.

(B) The value of the option is now found by recalling that the value of the option is equal to the value of the replicating portfolio. Plugging in numbers gives

$$C_E(0) = x(1)S(0) + y(1) = 1.2121.$$

(C) Using the risk-neutral valuation formula we find

$$\begin{aligned} C_E(0) &= \frac{1}{(1+R)} E_* \left[(S(1) - x)_+ \right] \\ &= \frac{1}{(1+R)} \left[p_*(S^u - X)_+ + (1 - p_*)(S^d - X)_+ \right]. \end{aligned}$$

Recall that $p_* = \frac{R-D}{U-D} = \frac{2}{3}$. Plugging in numbers and using basic calculations we again find

$$C_E(0) = 1.2121.$$

Problem 4

(A) The value of the European put is

$$\begin{aligned} P_E(0) &= \frac{1}{(1+R)} \left[p_*(X - S^u)_+ + (1 - p_*)(X - S^d)_+ \right] \\ &= \frac{1}{(1+R)} \left[\frac{R-D}{U-D}(1.1 - 1.2)_+ + \left(1 - \frac{R-D}{U-D} \right) (1.1 - 0.9)_+ \right] \\ &= \frac{1}{(1+0.1)} \left[0.666667 \cdot 0 + (1 - 0.666667) \cdot 0.2 \right] \\ &= 0.0606. \end{aligned}$$

(B) The value of the American put is

$$P_A(0) = max\{(X - S(0))_+; P_E(0)\}$$

so that

$$P_A(0) = max\{(1.1 - 1)_+; 0.0606\} = 0.1.$$

Problem 5

The put call-parity is

$$C_E(0) - P_E(0) = S(0) - Xe^{-rT}.$$

The following is one way of proving the put call-parity (there are also other reasonable ways).

Note that the payoff of one call and one short put is

$$max\{S(T) - X; 0\} - max\{X - S(T); 0\} = S(T) - X,$$

Hence, we can replicate (from the viewpoint of $t = 0$) the payoff of one call and one short put by having one share and loan corresponding to the amount Xe^{-rT} (which means we have to pay back X at T). Hence, by the no-arbitrage principle, the value of the replicating portfolio, $S(0) - Xe^{-rT}$, must be equal to the value of one call and one short put, which is $C_E(0) - P_E(0)$. The result follows.