MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Tentamensskrivning i
Ordinary differential equations VT 2023
7.5 hp
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No calculators, books, or other resources allowed. The total score is 24 points. The subsequent oral exam has a maximum of 6 points. An overall total of 15 points plus a successful completion of the group project are required to pass.

VERSION INCLUDING SOLUTIONS

Problem 1 (4 points)

Find all solutions to the differential equation $x'(t) + tx(t) = t^2 + 4t + 1$.

Solution: The ODE is linear, hence every solution can be uniquely written as the sum of a particular solution and a solution to the corresponding homogeneous equation. The homogeneous equation $x'(t) + tx(t) = 0$ is of order 1, hence its space of solutions is 1-dimensional. Moreover, we learnt the following formula for solutions in class:

$$x(t) = ce^{-\int_0^t \tau d\tau} = ce^{-t^2/2};\ c \in \mathbb{R}.$$

Alternatively, it could be solved by separating $x(t)$ and t.
To find a particular solution we use a polynomial Ansatz, since the right hand side of the equation is a polynomial and the left hand side is also a polynomial when x is. More precisely, in that case the left hand side is a polynomial of degree one larger than that of x. Hence x would have to be of degree 1 to be a solution, i.e., $x(t) = at + b$ and $x'(t) = a$. We arrive at the equation

$$a + t(at + b) = t^2 + 4t + 1,$$

which holds if and only if $a = 1$ and $b = 4$. In summary, the solutions are given by $x(t) = ce^{-t^2/2} + t + 4$ for all $c \in \mathbb{R}$.

Problem 2 (4 points)

Use the power series method to find the solution to the initial value problem:

$$\begin{aligned} x''(t) - 2tx'(t) + x(t) &= 0 \\ x(0) &= 1 \\ x'(0) &= 0 \end{aligned}$$

Solution: We assume that $x(t) = \sum_{k=0}^{\infty} a_k t^k$ solves the ODE and want to solve for the a_k. We have

$$x''(t) = \sum_{k=2}^{\infty} k(k-1)a_k t^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}t^k$$

and

$$tx'(t) = \sum_{k=1}^{\infty} ka_k t^{k-1} t = \sum_{k=0}^{\infty} ka_k t^k.$$

Hence, $x''(t) - 2tx'(t) + x(t)$ is again a power series with kth coefficient equal to

$$(k+2)(k+1)a_{k+2} - 2ka_k + a_k = (k+2)(k+1)a_{k+2} + (1-2k)a_k.$$

This expression equals 0 if and only if

$$a_{k+2} = \frac{2k-1}{(k+2)(k+1)} a_k.$$

This gives a recursive formula. Note that we are given $a_0 = x(0) = 1$ and $a_1 = x'(0) = 0$. The latter implies that $a_k = 0$ whenever k is an odd number. For $k = 2l$ even we obtain

$$a_k = \frac{\prod_{i=1}^{l}(4i-5)}{k!}.$$

PROBLEM 3 (4 POINTS)

Solve the following initial value problem via the Laplace transform:

$$x''(t) - x(t) = t$$
$$x(0) = 0$$
$$x'(0) = 0$$

(Hint: Recall that the Laplace transform of the function $f(t) = t$ equals $\frac{1}{s^2}$.)

Solution: For now we assume that x is a solution such that the Laplace transform L can be applied to x, x' and x''. Since $x(0) = 0 = x'(0)$ the formula for Laplace transforms of derivatives simplifies in this case to $L[x''] = s^2 L[x]$. We obtain the equation

$$s^2 L[x] - L[x] = \frac{1}{s^2}.$$

Solving for $L[x]$ gives

$$L[x] = \frac{1}{s^2(s^2-1)} = \frac{1}{s^2(s-1)(s+1)}.$$

We use partial fraction decomposition and want to find A, B, C, D such that

$$\frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s} + \frac{D}{s^2} = \frac{1}{s^2(s-1)(s+1)}.$$

This equation holds if and only if

$$A(s+1)s^2 + B(s-1)s^2 + C(s^2-1)s + D(s^2-1) = 1$$

or in other words

$$(A+B+C)s^3 + (A-B+D)s^2 - Cs - D = 1.$$

We first find that $D = -1$ and $C = 0$, and consecutively it follows that $B = -A$ and $2A = -1$. Hence

$$\frac{1}{2(s-1)} - \frac{1}{2(s+)1} - \frac{1}{s^2} = \frac{1}{s^2(s-1)(s+1)}.$$

We have $L^{-1}[\frac{1}{s-1} = e^t$ and $L^{-1}[\frac{1}{s+1} = e^{-t}$ by the shifting rule (and the fact that $L^{-1}[\frac{1}{s}] = 1$). Hence by linearity we obtain

$$L^{-1}[\frac{1}{s^2(s-1)(s+1)}] = \frac{1}{2}e^t - \frac{1}{2}e^{-t} - t.$$

The second deriviative of the function $x(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} - t$ equals $\frac{1}{2}e^t - \frac{1}{2}e^{-t}$, hence x solves the ODE. Moreover, $x(0) = 0$ and $x'(0) = \frac{1}{2} + \frac{1}{2} - 1 = 0$, so it also satisfies the initial conditions.

Problem 4 (4 points)

Find a fundamental matrix for the homogeneous system $x'(t) = Ax(t)$ with

$$A = \begin{pmatrix} 7 & 0 & -1 \\ 0 & 2 & 0 \\ 4 & 0 & 3 \end{pmatrix}.$$

Solution: For a linear homogeneous system with constant coefficients we know that a fundamental matrix is given by the matrix exponential $F(t) = e^{tA}$. To compute e^{tA}, we decompose $\mathbb{C}^3$ into generalized eigenspaces for A (and hence tA for any t).
First, we note that the second basis vector is an eigenvector for the eigenvalue 2, and that the span of the first and the third basis vector is preserved by A. Hence we can simplify and understand e^{tB} for the $2x2$-matrix

$$B = \begin{pmatrix} 7 & -1 \\ 4 & 3 \end{pmatrix}.$$

The characteristic polynomial of B is given by

$$(\lambda - 7)(\lambda - 3) + 4 = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2.$$

Hence B has only one complex eigenvalue, namely 5, and the generalized eigenspace is all of $\mathbb{C}^2$. Since B is not diagonal, it follows that the eigenspace must be 1-dimensional. For example the second basis vector $e_2 = (0, 1)$ is not an eigenvector. It follows that
$(B - 5\mathbb{1})e_2 = -e_1 - 2e_2 = (-1, -2)$ is an eigenvector, and that expressed in the basis $((-1, -2), (0, 1))$ the matrix B takes the form

$$C = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}.$$

Hence we have $B = T^{-1}CT$, where

$$T = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$$

which is self-inverse, i.e., $T = T^{-1}$. Furthermore, we have

$$e^{tC} = e^{5t\mathbb{1}}e^{tC-5t\mathbb{1}} = \begin{pmatrix} e^{5t} & te^{5t} \\ 0 & e^{5t} \end{pmatrix}.$$

Therefore,

$$e^{tB} = Te^{tC}T = \begin{pmatrix} e^{5t} & te^{5t} \\ 0 & e^{5t} \end{pmatrix} = \begin{pmatrix} e^{5t} + 2te^5t & -te^{5t} \\ 4te^{5t} & -2te^{5t} + e^{5t} \end{pmatrix}.$$

Combining this with the matrix exponential of A restricted to the second basis vector (which is simply e^{2t}), we finally obtain

$$F(t) = e^{tA} = \begin{pmatrix} e^{5t} + 2te^{5}t & 0 & -te^{5t} \\ 0 & e^{2t} & 0 \\ 4te^{5t} & 0 & -2te^{5t} + e^{5t} \end{pmatrix}.$$

Problem 5 (4 points)

We consider the boundary conditions $y(0) = y(1), y'(0) = 0$ on the interval $[0, 1]$.

(1) Give an example (with proof) of a 2nd order linear ODE for which the associated boundary value problem has a unique solution.
(2) Give an example (with proof) of a 2nd order linear ODE for which the associated boundary value problem does not have a unique solution.

Solution: (1): This is the 'regular' case for which most linear ODEs will work. We consider the linear ODE with constant coefficients $y'' - y = 0$. The eigenvalues of the characteristic polynomial are 1 and -1, hence the space of solutions is spanned by $y_1(x) = e^x$ and $y_2(x) = e^{-x}$. A general solution is therefore of the form

$$y(x) = ae^x + be^{-x},$$

with derivative $y'(x) = ae^x - be^{-x}$. Assuming $y'(0)$ yields $a = b$. Further assuming $y(0) = y(1)$ then gives $2a = a(e + e^{-1})$. As $e + e^{-1} \neq 2$, this forces $a = b = 0$.
(2): We consider the ODE $y'' = 0$, whose solutions are precisely the degree 1 polynomials $y(x) = ax + b$. Of these, all constant functions $y(x) = b$ satisfy the boundary conditions, hence the solution is not unique.
An alternative approach is to first take a non-zero function y satisfying the boundary conditions and then constructing the ODE so that y is a solution. For example, the function $y(x) = \cos(2\pi x)$ is periodic with $y'(0) = 0$ and gives a solution of the linear ODE $y'' + 4\pi^2 y = 0$.

Problem 6 (4 points)

Consider the autonomous system

$$\begin{cases} x' = -3x + 5y \\ y' = (x - y)(\cos(x + y) - 2) \end{cases}$$

What are its equilibrium points? Determine for each of the equilibrium points wheter it is stable, asymptotically stable or unstable.
Solution: Note that $\cos(x + y) - 2$ is never 0, hence (x, y) is an equilibrium point if and only if $-3x + 5y = 0 = x - y$. The second equation forces $x = y$, and then the first forces $x = y = 0$. Hence there is a single equilibrium point $(0, 0)$. To study its stability we note that the autonomous system is everywhere differentiable with derivative

$$\begin{pmatrix} -3 & 5 \\ (\cos(x + y) - 2) + (x - y)(-\sin(x + y)) & -(\cos(x + y) - 2) + (x - y)(-\sin(x + y)) \end{pmatrix}$$

For $(x, y) = (0, 0)$ we obtain the matrix

$$A = \begin{pmatrix} -3 & 5 \\ -1 & 1 \end{pmatrix}.$$

Its characteristic polynomial equals $p_A(\lambda) = (-3 - \lambda)(1 - \lambda) + 5 = \lambda^2 + 2\lambda + 2$, with roots (and hence eigenvalues for A) given by $\lambda_1 = i - 1$ and $\lambda_2 = -i - 1$. As the real parts of both λ_1 and λ_2 are negative, a theorem from the last course of the lecture implies that $(0, 0)$ is asymptotically stable both for the autonomous system $x' = Ax$ and the original autonomous system

$$\begin{cases} x' = -3x + 5y \\ y' = (x - y)(\cos(x + y) - 2). \end{cases}$$