Solution logic exam 2023-08-09

① These are theory questions and we refer to the literature

② (a) $\neg P_2$ (b) $= \neg P_1$ (c) $= P_1$

③ By using induction we have that

(a) $FV(\varphi) = FV\left(\forall x_1 (P_1(x_1) \wedge P_1(x_3)) \to \exists x_2 (f_1(x_2) \doteq x_1)\right) \setminus \{x_3\}$

$$\cup$$

$$FV\left(P_1(x_1) \vee P_1(f_1(x_2))\right) \setminus \{x_1\}$$

$$= \left(\{x_1, x_3\} \setminus \{x_1\} \cup \{x_1, x_2\} \setminus \{x_2\}\right) \setminus \{x_3\}$$

$$\cup$$

$$\{x_1\, x_2\} \setminus \{x_1\}$$

$$= (\{x_3\, x_1\} \setminus \{x_3\} \cup \{x_2\} = \{x_1, x_2\}$$

(b)

$\varphi[x_1/x_2] = \exists x_3 (\forall x_1 (P(x_1) \wedge P_1(x_3)) \to \exists x_2 (f_1(x_2) \doteq x_1))$

$\wedge \forall x_1\, P_1(x_1) \vee P_1(f_1(x_1))$

$\varphi[t/x_3] = \varphi$ since $x_3 \notin FV(\varphi)$

$\varphi[f_1(x_3)/x_1] = \exists x_3 (\forall x_1 (P_1(x_1) \wedge P_1(x_3) \to$

$\to \exists x_2\, f_1(x_2) \doteq \cancel{x_1}\, f_1(x_3)) \to$

$\forall x_1\, P(x_1) \vee P(f_1(x_2))$

x_1 is not free for x_2 in φ
every term is free for x_3 in φ
$f_1(x_3)$ is free for x_1 in φ

(d)

$$[\![\varphi]\!]^{\mathcal{A}\sigma} = [\![\exists x_3 \, (\forall x_1 \, (P(x_1) \wedge P(x_3) \rightarrow \exists x_2 (f_1(x_2) = x_1)) \wedge [\forall x_1 \, P(x_1) \vee P(f_1(x_2))]]\!]^{\mathcal{A}\sigma}$$

Let us look at the second subformula

$$[\![P(x_1) \vee P(f_1(x_2))]\!]^{\mathcal{A}\,\sigma(x_1 \mapsto a)}$$

$$= [\![a \in E]\!] \vee [\![1 \in E]\!]$$

↳ false.

↳ true only if $a \in E$

$\Rightarrow$ This is false so φ cannot be true in this interpretation

(e) $\sigma(x_2) = 1$ makes the second subformula true for all x_1

we have thus to ~~find~~ assign values to x_1 for which the first is true

Observe that
for $x_3 = 1$ we have that $[\![P(x_1) \wedge P(x_3)]\!] = 0$ for all x_1
in particular
$$[\![\forall x_1 \, P(x_1) \wedge P(x_3)]\!] = 0$$

we deduce that

$$[\![\forall x_1\, P(x_1) \wedge P(x_3) \to \exists x_2\, f_1(x_2) \doteq x_1]\!]^{A\,\omega(x_3 \to 1)} = 1$$

in particular we have that

$$[\![\exists x_3 (\forall x_1 \;\cdots\; \to \;\cdots\;)]\!]^{A\omega} = 1$$

Thus the evaluation

$\omega = v[x_2 \mapsto 1]$ makes the formula true

(4)

(a)

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg P_1]^1 \quad \neg P_1 \to \neg P_2}{\neg P_2}\to E \quad [P_2]^2}{\bot}\to E}{P_1}\text{RAA }1}{P_2 \to P_1}\to I_2$$

(b)

$$\dfrac{\dfrac{\exists x_1 P_1(x_1) \wedge P_1(f_1)}{\exists x_1 P_1(x_1)}\wedge E \qquad \dfrac{\dfrac{\dfrac{\exists x_1 P_1(x_1) \wedge P(f_1)}{P(f_1)}\wedge E \quad [P_1(x_1)]^1}{P_1(x_1) \wedge P_1(f_1)}\wedge I}{\exists x_1 (P(x_1) \wedge P_1(f_1))}\exists I}{\exists x_1 (P(x_1) \wedge P(f_1))}\exists E

5) $\forall x_1 \forall x_2 \; f(x_1) \doteq f(x_2) \longrightarrow x_1 \doteq x_2$ - φ_{in}

(a) $\forall x_1 \forall x_2 \neg(x_1 \doteq x_2) \longrightarrow \neg(f_1(x_1) \doteq f_1(x_2))$ = φ_{in}

$\forall x_1 \exists x_2 \; f_1(x_2) \doteq x_1$ $\quad \varphi_{onto}$

(b) This theory is not consistent

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x_0 \; f_1(f_1(x_0)) \doteq x_0}{f_1(f_1(x_1)) \doteq x_1} \forall E}{\exists x_2 \; f_1(x_2) \doteq x_1} \exists I}{\forall x_1 \exists x_2 \; f_1(x_2) \doteq x_1} \forall I \qquad \neg\varphi_{onto}}{\perp}$$

$= (f_1(x_2) \doteq 1)[f_1(x_1)/x]$

Semantic argument:
If there were a countermodel then there would be a non-~~injective~~ surjective involution. But involutions have inverse so they are bijective and hence surjective

(c) as φ_{in} we use
$\forall x_1 \forall x_2 \; f_1(x_1) \doteq f_1(x_2) \longrightarrow x_1 \doteq x_2$

$\longrightarrow$

$$\frac{\forall x_0\ f_1 f_1(x_0) \doteq x_0}{f_1 f_1(x_1) \doteq x_1}\ \forall E$$

$$[f_1(x_1) \doteq f_1(x_2)]^1$$

$$\frac{f_1 f_1(x_1) \doteq x_1 \qquad [f_1(x_1) \doteq f_1(x_2)]^1}{f_1(f_1(x_2)) \doteq x_1}\ \text{REP}$$

$$\frac{\forall x_0\ ff(x_0) \doteq x_0}{ff(x_2) = x_2}$$

$$\frac{f_1(f_1(x_2)) \doteq x_1 \qquad ff(x_2) = x_2}{x_1 \doteq x_2}\ \text{REP}$$

$$\frac{x_1 \doteq x_2}{f(x_1) \doteq f(x_2) \longrightarrow x_1 \doteq x_2}\ \to I\ 1$$

$$\frac{f(x_1) \doteq f(x_2) \longrightarrow x_1 \doteq x_2}{\forall x_2\ f_1(x_1) \doteq f_1(x_2) \to x_1 \doteq x_2}\ \forall I$$

$$\frac{\forall x_2\ f_1(x_1) \doteq f_1(x_2) \to x_1 \doteq x_2}{\forall x_1\ \forall x_2\ f_1(x_1) \doteq f_1(x_2) \to x_1 = x_2}\ \forall I$$

(6)

$$\neg(x_1 \doteq f_2) \wedge \forall x_2 \forall x_3 \left(f_4(x_2\, x_3) \doteq x_1 \longrightarrow x_2 \doteq f_2 \vee x_3 \doteq f_1 \right)$$

(7) (a) this is ~~possible~~ not possible.

Let $\varphi = P_1(x_1)$ $\mathcal{A} = \langle A, \emptyset \rangle$

Then $\forall x_1 \neg\varphi$ is true since all elements of a do not belong to $\emptyset$

On the other side $\exists x_1 P_1(x_1)$ is false since no elements belongs to the empty set

(a) $\varphi = P_1(x_1)$ $\psi = P_2(x_3)$

$\mathcal{A} = (\mathbb{R}, \mathbb{R}^{\geq 0}, \mathbb{R}^-)$

both $\mathbb{R}^{\geq 0}$ and $\mathbb{R}^-$ are non empty so $\exists x_1 \varphi$ and $\exists x_3 \psi$ are true but $\mathbb{R}^{\geq 0} \cap \mathbb{R}^- = \emptyset$ so

$\exists x_1\ \varphi \wedge \psi$ is false

(8) (a) The answer is yes. to see this we have to show that Γ' is consistent and that if $\Gamma \cup \{\varphi\}$ is consistent then $\varphi \in \Gamma'$

Γ' is consistent:
suppose otherwise that $\exists\ \gamma_1 \dots \gamma_n \in \Gamma'$
with $\gamma_1 \dots \gamma_n \vdash \perp$
Since $\mathbb{Q}\,v \models \gamma_1 \dots \gamma_n$ by soundness
we would have $\mathbb{Q}\,v \models \perp$ which is
a contradiction

Γ' is maximally consistent
Let $\varphi \in Form$ such that $\mathbb{Q}\,v \not\models \varphi$
then $\mathbb{Q}\,v \models \neg\varphi$ ~~by completeness~~
so $\neg\varphi \in \Gamma'$
and so $\Gamma' \cup \{\varphi\} \vdash \perp$

thus is
$\Gamma' \cup \{\varphi\} \nvdash \perp$ we have that
$\mathbb{Q}\,v \vdash \varphi$ and so
$\varphi \in \Gamma'$.

(b) Γ' does not have the existence property

Consider the formula
$\exists x_1 \neg (R(x_0\ x_0)) = \exists x_1 \varphi$

This "translates" in there is a rational number
$q < v(x_0) = 0$
which is true so $\exists x_1 \varphi \in T_{\mathbb{Q}v}$

On the other side $\mathbb{Q}\,v$ does not believe
any $\varphi[t/x_1]$

to - Since there are no functions
Term = Var
let $t = x_j$

$\varphi[t/x_1] = \neg P(x_0, x_j)$

which is true only if
$j = v(x_j) < v(x_0) = 0$

which never happens

So $\mathfrak{A}v \not\models \varphi$ and the theory does not have the existence property