MT7001 – Probability theory III – exam

Date Monday December 12, 2022
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Tools None.
Grading criteria The exam consists of two parts, which consist of 20 and 40 points respectively. To pass the exam a score of 14 or higher is required on Part I. If attained, then also Part II is graded, and the score on this part determines the grade. Grades are determined according to the following table:

	A	B	C	D	E
Part I	14	14	14	14	14
Part II	32	24	16	8	0

Problems of Part I may give up to five points each, and problems of Part II may give up to ten points each. Complete and well motivated solutions are required for full score. Partial solution may be rewarded with a partial score.

Part I

Problem 1. Let $\Omega = \{1,2,3,4,5,6\}$. Determine which of the following collections of subsets of Ω that are σ-algebras on Ω:

$\mathcal{F}_1 = \big\{\emptyset, \{1,2,3\}, \{4,5\}, \{1,2,3,4,5\}\big\}$,
$\mathcal{F}_2 = \big\{\emptyset, \{1,2,3\}, \{4,5,6\}, \{1,2,3,4,5,6\}\big\}$,
$\mathcal{F}_3 = \big\{\emptyset, \{1\}, \{2\}, \{3,4,5,6\}, \{1,2,3,4,5,6\}\big\}$,
$\mathcal{F}_4 = \big\{\emptyset, \{1\}, \{2,3,4,5\}, \{6\}, \{1,2,3,4,5\}, \{1,6\}, \{2,3,4,5,6\}, \{1,2,3,4,5,6\}\big\}$.

Problem 2. Consider the probability space $([0,1], \mathcal{B}, \mathbb{P})$, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,1]$ and $\mathbb{P}$ denotes Lebesgue measure. Show that $\mathbb{P}(X = Z) \neq 1$, but that X and Z have the same distribution, where

$$X(\omega) = \begin{cases} \omega & \text{for } \omega \in [0,1/2), \\ 1/2 & \text{for } \omega \in [1/2,1]; \end{cases} \qquad Z(\omega) = \begin{cases} \omega & \text{for } \omega \in [0,1/4), \\ 1/2 & \text{for } \omega \in [1/4,3/4), \\ 5/4-\omega & \text{for } \omega \in [3/4,1]. \end{cases}$$

Problem 3. Let $X_1, X_2, \dots$ be i.i.d. random variables with zero mean and variance one. Let $S_n = X_1 + \ldots + X_n$, and show that, as $n \to \infty$, we have

$$\frac{S_n}{\sqrt{n \log\log n}} \to 0 \quad \text{in probability.}$$

Problem 4. Let $X_1, X_2, \dots$ be i.i.d. random variables with mean zero and variance one. Set $S_0 = 0$, and let $S_n = X_1 + \ldots + X_n$ and $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ for $n \geq 1$. Prove that $(S_n)_{n\geq 0}$ and $(S_n^2 - n)_{n\geq 0}$ are martingales with respect to $(\mathcal{F}_n)_{n\geq 0}$.

Part II

Problem 5. Let $(X_n)_{n\geq 1}$ be a square-integrable martingale (i.e. $\mathbb{E}[X_n^2] < \infty$ for all $n \geq 1$) with respect to itself.

(a) Show that $\mathbb{E}[X_{n+1} - X_n] = 0$ for all $n \geq 1$.

(b) Show that the increments of $(X_n)_{n\geq 1}$ are uncorrolated, i.e. that

$$\mathrm{Cov}(X_{i+1} - X_i, X_{j+1} - X_j) = 0 \quad \text{for } i \neq j.$$

(c) Prove that $\mathbb{E}[X_n^2]$ is increasing as a function of n.

Problem 6. Let $X_1, X_2, \ldots$ be independent exponentially distributed random variables with intensity one, i.e. whose distribution function is given by $F(x) = 1 - e^{-x}$ for $x \geq 0$. Let $N_n = \#\{1 \leq k \leq n : X_k > \log k\}$ and set $N = \lim_{n\to\infty} N_n$. (Here and elsewhere log denotes the natural logarithm.)

(a) Prove that $\mathbb{E}[N] = \infty$.

(b) Prove that $\mathbb{P}(N = \infty) = 1$.

(c) Prove that $N_n / \mathbb{E}[N_n] \xrightarrow{p} 1$ as $n \to \infty$.

Hint: One may show that $\mathrm{Var}(N_n) \leq \mathbb{E}[N_n]$.

Problem 7. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω consists of all sequences $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$ of zeros and ones of length four, $\mathcal{F}$ denotes the collection of all subsets of Ω, and $\mathbb{P}$ denotes the measure that assigns equal probability to the elements of Ω.

(a) Determine the σ-algebra generated by $X(\omega) = \omega_1 + \omega_2 + \omega_3 + \omega_4$.

(b) Using the definition of expectation, determine $\mathbb{E}[X]$.

Problem 8. Let $X_1, X_2, \ldots$ be independent random variables taking values -1 and 1 with equal probability. Set $S_0 = 0$, and $S_n = X_1 + \ldots + X_n$ for $n \geq 1$. Let $m \geq 1$ be an integer and define

$$T = \min\{n \geq 1 : |S_n| = m\} \quad \text{and} \quad T' = \min\{n \geq 1 : S_n = m\}.$$

(a) Show that T and T' are stopping times with respect to $(\mathcal{F}_n)_{n\geq 1}$, where $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

(b) Prove that $\mathbb{E}[T] < \infty$ and $\mathbb{E}[T'] = \infty$.