Exam Probability theory III 2022-12-12

PART I

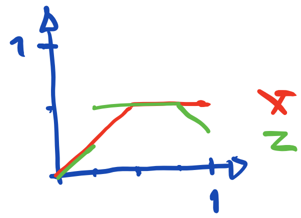
$\boxed{1}$ Given a set Ω, a collection $\mathcal{F}$ of subsets of Ω is a σ-algebra on Ω if

- $\Omega \in \mathcal{F}$
- $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
- $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$. 2p

$\mathcal{F}_2$ and $\mathcal{F}_4$ both satisfy these conditions, and are σ-algebras on $\Omega = \{1,2,3,4,5,6\}$.

$\mathcal{F}_1$ does not contain Ω and $\mathcal{F}_3$ does not contain e.g. $\{1\} \cup \{2\} = \{1,2\}$. These are then not σ-algebras on Ω. 3p

$\boxed{2}$ We first draw a figure.



It is clear from the figure and the definition that

$$\{X = Z\} = [0, 1/4) \cup [1/2, 3/4].$$

$$\mathbb{P}(X=Z) = \mathbb{P}([0,1/4)) + \mathbb{P}([1/2, 3/4]) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \neq 1.$$

2p

We also have

$$\{X \le x\} = \begin{cases} [0,x] & x < 1/2 \\ [0,1] & x \ge 1/2 \end{cases}$$

$$\mathbb{P}(X \le x) = \begin{cases} x & 0 \le x < 1/2 \\ 1 & x \ge 1/2 \end{cases}$$

and

$$\{Z \le x\} = \begin{cases} [0,x] & x < 1/4 \\ [0,x] \cup [5/4 - x, 1] & 1/4 \le x < 1/2 \\ [0,1] & x \ge 1/2 \end{cases}$$

$$\mathbb{P}(Z \le x) = \begin{cases} x & x < 1/4 \\ \frac{1}{4} + (x - \frac{1}{4}) = x & 1/4 \le x < 1/2 \\ 1 & x \ge 1/2 \end{cases}$$

Hence $F_X = F_Z$ so X and Z have the same distribution. 3p

3 Let X_1, X_2 be iid zero mean, variance one. Set $S_n = X_1 + X_2 + \dots + X_n$. Then $\mathbb{E}[S_n] = 0$, so by Chebyshev's inequality

$$\mathbb{P}\left(|S_n| \geq \varepsilon \cdot \sqrt{n \log\log n} \right) \leq \frac{\mathrm{Var}(S_n)}{\varepsilon^2 n \log\log n}. \qquad 2p$$

This holds $\forall \varepsilon > 0$. Since iid variance one,

$$\mathrm{Var}(S_n) = \sum_{k=1}^n \mathrm{Var}(X_k) = n.$$

This gives

$$\mathbb{P}\left(\frac{|S_n|}{\sqrt{n \log\log n}} \geq \varepsilon \right) \leq \frac{1}{\varepsilon^2 \log\log n} \to 0 \qquad 3p$$

as $n \to \infty$. Since true for all $\varepsilon > 0$, this shows conv. in probab.

4 Let $X_1, X_2, \dots$ be iid zero mean, variance one. Set $S_n = X_1 + \dots + X_n$ and $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

Clearly both S_n and $S_n^2 - n$ are $\mathcal{F}_n$-measurable since functions of $X_1, \dots, X_n$. Also

$$\mathbb{E}[|S_n|] \leq \sum_{k=1}^n \mathbb{E}[|X_k|] \leq \sum_{k=1}^n \mathbb{E}[X_k^2]^{1/2} = n.$$

$$\mathbb{E}[|S_n^2 - n|] \leq \mathbb{E}[S_n^2] + n = \sum_{k=1}^n \mathbb{E}[X_k^2] + n = 2n. \qquad 2p$$

Finally we verify that the two sequences satisfy the martingale property.

$$\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] = S_n + \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = S_n.$$

$$\begin{aligned}
\mathbb{E}[S_{n+1}^2 - (n+1) \mid \mathcal{F}_n] &= \mathbb{E}[(S_n + X_{n+1})^2 - (n+1) \mid \mathcal{F}_n] \\
&= \mathbb{E}[S_n^2 + 2 S_n \cdot X_{n+1} + X_{n+1}^2 \mid \mathcal{F}_n] - (n+1) \\
&= S_n^2 + 2 S_n \underbrace{\mathbb{E}[X_{n+1} \mid \mathcal{F}_n]}_{=0} + \underbrace{\mathbb{E}[X_{n+1}^2 \mid \mathcal{F}_n]}_{=1} - (n+1) \\
&= S_n^2 - n.
\end{aligned}$$

So $(S_n)_{n \geq 0}$ and $(S_n^2 - n)_{n \geq 0}$ are martingales wrt $(\mathcal{F}_n)_{n \geq 0}$. 3p

$\boxed{\text{PART II}}$

$\boxed{5}$ Let $(X_n)_{n\geq 1}$ be a square-integrable martingale wrt $(\mathcal{F}_n)_{n\geq 1}$ where $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

(a) We have for $n \geq 1$ that

$$\mathbb{E}[X_{n+1} - X_n] = \mathbb{E}\big[\underbrace{\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n]}_{=0 \text{ since martingale}}\big] = 0.$$

3p

(b) Suppose $i > j$. Then

$$\mathrm{Cov}(X_{i+1} - X_i, X_{j+1} - X_j)$$

$\mathbb{E}[X_{k+1} - X_k] = 0$ from (a)

$$= \mathbb{E}\big[(X_{i+1} - X_i)(X_{j+1} - X_j)\big]$$

$i \geq j+1$

$$= \mathbb{E}\big[(X_{j+1} - X_j)\, \underbrace{\mathbb{E}[X_{i+1} - X_i \mid \mathcal{F}_i]}_{=0}\big]$$

$$= 0$$

3p

(c) For $n \geq 1$ we have

$$X_{n+1}^2 = (X_n + (X_{n+1} - X_n))^2$$

$$= X_n^2 + \underbrace{(X_{n+1} - X_n)^2}_{\geq 0} + 2X_n(X_{n+1} - X_n).$$

This gives

$$\mathbb{E}[X_{n+1}^2] \geq \mathbb{E}[X_n^2] + 2\,\mathbb{E}[X_n(X_{n+1} - X_n)]$$

$$= \mathbb{E}[X_n^2] + 2\,\mathbb{E}\big[X_n\, \underbrace{\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n]}_{=0}\big]$$

$$= \mathbb{E}[X_n^2]$$

Hence $\mathbb{E}[X_{n+1}^2] \geq \mathbb{E}[X_n^2]$ for all $n \geq 1$.

4p

$\boxed{6}$ Let $X_1, X_2, \ldots$ be independent $\exp(1)$-distributed, and set $A_k = \{X_k > \log k\}$. Note that

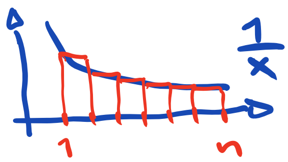
$$N_n = \sum_{k=1}^{n} \mathbb{1}_{A_k} \quad \text{and} \quad N = \sum_{k=1}^{\infty} \mathbb{1}_{A_k}.$$

(a) Clearly $N_n \le N$ for all $n \ge 1$. Hence, for all $n \ge 1$

$$\mathbb{E}[N] \ge \mathbb{E}[N_n] = \sum_{k=1}^{n} \mathbb{P}(A_k) = \sum_{k=1}^{n} \frac{1}{k}.$$

Since

$$\sum_{k=1}^{n} \frac{1}{k} \ge \int_1^n \frac{1}{x}\,dx = \log n \nearrow \infty$$



as $n \to \infty$, the claim follows. 3p

(b) The above computation and Borel-Cantelli's second lemma give (events independent)

$$\mathbb{P}(N = \infty) = \mathbb{P}(A_k \text{ i.o.}) = 1.$$ 3p

(c) Chebyshev's inequality gives

$$\mathbb{P}(|N_n - \mathbb{E}[N_n]| > \varepsilon\,\mathbb{E}[N_n]) \le \frac{\mathrm{Var}(N_n)}{\varepsilon^2\,\mathbb{E}[N_n]^2}.$$ 2p

Moreover, due to independence

$$\mathrm{Var}(N_n) = \sum_{k=1}^{n} \mathrm{Var}(\mathbb{1}_{A_k}) \le \sum_{k=1}^{n} \mathbb{P}(A_k) = \mathbb{E}[N_n].$$

From (a) we conclude that, as $n \to \infty$,

$$\mathbb{P}(|N_n - \mathbb{E}[N_n]| > \varepsilon\,\mathbb{E}[N_n]) \le \frac{1}{\varepsilon^2\,\mathbb{E}[N_n]} \le \frac{1}{\varepsilon^2 \log n} \to 0.$$

This shows that $\frac{N_n}{\mathbb{E}[N_n]} \xrightarrow{P} 1$. 2p

$\boxed{7}$ $\Omega = \{0,1\}^4$, $\mathcal{F}$ the power set of Ω and $\mathbb{P}$ the uniform measure on Ω.

(a) The random variable X takes the values $0,1,2,3,4$. For $k=0,1,\dots,4$ let

$$A_k = \{\omega : X(\omega) = k\}.$$

Then $\mathcal{P} = \{A_0, A_1, A_2, A_3, A_4\}$ is a partition of Ω. $\sigma(X)$ contains all sets that can be written as a union of elements in $\mathcal{P}$. There are $2^5 = 32$ sets in $\sigma(X)$. 5p

(b) X is a simple rv since it only takes finitely many values. By definition we have

$$\mathbb{E}[X] = \sum_{k=0}^{4} k \cdot \mathbb{P}(A_k).$$

Since $\mathbb{P}$ is the uniform measure, we have

$$\mathbb{P}(A_k) = \frac{|A_k|}{|\Omega|} = \frac{\binom{4}{k}}{2^4}.$$ 5p

Hence

$$\mathbb{E}[X] = \frac{1}{2^4}\left(0 + 1\cdot 4 + \overset{k}{2} \cdot \overset{\binom{4}{k}}{6} + 3\cdot 6 + 4 \cdot 1\right)$$

$$= \frac{38}{16} = \frac{19}{8}.$$

$\boxed{8}$ (a) Clearly S_n is $\mathcal{F}_n$-measurable, since a function of $X_1, \dots, X_n$. Hence T and T' are stopping times since

$$\{T=n\} = \{S_1 \neq \pm m\} \cap \dots \cap \{S_{n-1} \neq \pm m\} \cap \{S_n = \pm m\},$$

$$\{T'=n\} = \{S_1 \neq m\} \cap \dots \cap \{S_{n-1} \neq m\} \cap \{S_n = m\},$$

and these are events in $\mathcal{F}_n$. 3p

(b) Let us first show that $\mathbb{E}[T] < \infty$. For $k \geq 1$ let

$$A_k = \{X_k = X_{k+1} = \ldots = X_{k+2m} = 1\},$$

and note that A_k implies that starting at step k, the sequence $(S_n)_{n \geq 1}$ takes $2m+1$ steps up in a row. Hence if $|S_{k-1}| < m$ and A_k occurs, then $S_{k+2m} \geq m$. Let $N = \min\{k \geq 1 : A_k \text{ occurs}\}$. Then

$$T \leq N + 2m.$$

Next note that A_k and $A_{k+\ell}$ are independent if $\ell \geq 2m+1$. Hence let $N' = \min\{k \geq 1 : A_{(2m+1)k} \text{ occurs}\}$ and note that

$$N \leq (2m+1) N'.$$

4p

Moreover, N' is geometrically distributed with parameter $p = 1/2^{2m+1}$, and hence that $\mathbb{E}[N'] < \infty$. So,

$$\mathbb{E}[T] \leq \mathbb{E}[N] + 2m \leq (2m+1)\mathbb{E}[N'] + 2m < \infty.$$

We next show that $\mathbb{E}[T'] = \infty$. Suppose the contrary. Then since $(S_n)_{n \geq 0}$ is a martingale with bounded increments, it follows from Optional stopping III that

$$\mathbb{E}[S_{T'}] = \mathbb{E}[S_0] = 0.$$

3p

However $\mathbb{E}[S_{T'}] = m$, which contradicts the assumption that $\mathbb{E}[T'] < \infty$. $\lightning$