MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Jonathan Rohleder

Tentamensskrivning i
Ordinary differential equations VT 2023
7.5 hp
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No calculators, books, or other resources allowed. The total score is 24 points. The subsequent oral exam has a maximum of 6 points. An overall total of 15 points plus a successful completion of the group project are required to pass.

Problem 1 (4 points)

Find the (unique) solution to

$$x''(t) + 2x'(t) - 15x(t) = 30t + 11$$

satisfying the initial values $x(0) = 1$ and $x'(0) = -4$.

Solution: We start by solving the associated homogeneous differential equation, which is linear with constant coefficients of degree 2. Hence we consider its characteristic polynomial $\lambda^2 + 2\lambda - 15$, whose roots are given by 3 and -5 (they can be found by first checking the factors of the constant part -15). Hence the space of solutions is spanned by the function e^{3t} and e^{-5t}. Next we find one specific solution to the inhomogeneous equation, for which we apply the polynomial approach. Since $30t + 11$ is of degree one and the degree of $x''(t) + 2x'(t) - 15x(t)$ will equal that of x (assuming we choose x to be a polynomial), we are aiming for a degree one polynomial. Let $x(t) = at + b$. Then $x''(t) = 0$ and $x'(t) = a$, hence

$$x''(t) + 2x'(t) - 15x(t) = 2a - 15at - 15b = -15at + (2a - 15b).$$

Setting this equal to $30t + 11$ yields $a = -2$ and $b = -1$. Hence, $-2t - 1$ is a solution, and the general solution is of the form

$$x_{\alpha,\beta}(t) = -2t - 1 + \alpha e^{3t} + \beta e^{-5t}$$

for $\alpha, \beta \in \mathbb{R}$. It remains to find α and β so that the initial values are satisfied. We have

$$x_{\alpha,\beta}(0) = -1 + \alpha + \beta$$

and

$$x'_{\alpha,\beta}(0) = -2 + 3\alpha - 5\beta.$$

Hence, we need $\alpha + \beta$ to equal 2, and $3\alpha - 5\beta$ to equal -2, which is satisfied precisely by $\alpha = \beta = 1$. In summary, the unique solution is given by

$$x_{1,1}(t) = -2t - 1 + e^{3t} + e^{-5t}.$$

Problem 2 (4 points)

Use the Laplace transform to find the solution to the following initial value problem:

$$\begin{aligned} x''(t) - 3x'(t) + 2x(t) &= e^{3t} \\ x(0) &= 1 \\ x'(0) &= 0 \end{aligned}$$

Solution: Let $L[-](s)$ denote the Laplace transform. Applied to a potential solution x to the ODE we obtain the equation

$$L[x''(t) - 3x'(t) + 2x(t)](s) = (s^2 L[x](s) - s) - 3(sL[x](s) - 1) + 2L[x](s) = \frac{1}{s-3} = L[e^{3t}](s),$$

where we used the given initial values of $x(0)$ and $x'(0)$ and standard properties of the Laplace transform. This shows that

$$L[x](s) = \frac{s^2 - 6s + 10}{(s-3)(s^2 - 3s + 2)} = \frac{s^2 - 6s + 10}{(s-3)(s-1)(s-2)}.$$

Partial fractions let us rewrite this as

$$L[x](s) = \frac{1}{2(s-3)} + \frac{5}{2(s-1)} - \frac{2}{s-2}.$$

Applying the inverse Laplace transform we find that

$$x(t) = \frac{1}{2}e^{3t} + \frac{5}{2}e^{t} - 2e^{2t}.$$

A quick check then shows that this function indeed solves the given initial value problem.

Problem 3 (4 points)

Find a fundamental matrix for the homogeneous system $x'(t) = Ax(t)$ with

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}.$$

Solution: We know that a fundamental matrix is given by the matrix exponential e^{At}. To compute this, we transform A into a slightly simpler matrix. First of all, note that the first and third basis vector already span a Jordan block for the eigenvalue 3. Since the determinant of A equals 18 it follows that the other eigenvalue must equal 2. Solving for an eigenvector yields $(1, 1, -1)$ as a possible choice. Hence, expressed in this basis (i.e., $(1, 0, 0), (0, 0, 1), (1, 1, -1)$) the matrix exponential takes the form

$$\begin{pmatrix} e^{3t} & te^{3t} & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}.$$

It remains to transform back to the original basis, i.e., to multiply with the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

from the left and with its inverse

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

from the right. Multiplied out we obtain

$$e^{At} = \begin{pmatrix} e^{3t} & -e^{3t} + te^{3t} + e^{2t} & te^{3t} \\ 0 & e^{2t} & 0 \\ 0 & e^{3t} - e^{2t} & e^{3t} \end{pmatrix}.$$

Problem 4 (4 points)

Consider the following boundary problem. Compute its associated Green's function and express the solution in terms of it.

$$\begin{aligned} y''(x) - y(x) &= x^4 + 1 \\ y(0) &= 0 \\ y(1) &= 0 \end{aligned}$$

Solution: The boundary conditions are separable, hence Green's function can be determined via the Wronskian function. For this we have to find a non-zero twice differentiable function $y_1(x)$ satisfying $y_1''(x) - y_1(x) = 0$ and $y(0) = 0$, as well as a function $y_2(x)$ satisfying the same differential equation and $y_2(1) = 0$. The solution space to $y''(x) - y(x) = 0$ is spanned by the functions e^x and e^{-x}. It follows that we can take

$$y_1(x) = e^x - e^{-x},$$

and

$$y_2(x) = e^x - e^2 e^{-x} = e^x - e^{2-x}.$$

The wronskian $w(x) = y_1 y_2' - y_2 y_1'$ hence computes as

$$w(x) = (e^x - e^{-x})(e^x + e^{2-x}) - (e^x - e^{2-x})(e^x + e^{-x}),$$

which multiplies out to

$$w(x) = e^{2x} + e^2 - 1 - e^{2-2x} - e^{2x} - 1 + e^2 + e^{2-2x} = 2(e^2 - 1).$$

Hence, Green's function is given by

$$G(x, \xi) = \begin{cases} \frac{y_1(x) y_2(\xi)}{w(\xi)} = \frac{(e^x - e^{-x})(e^\xi - e^{2-\xi})}{(2e^2 - 2)} & 0 \leq x < \xi \\ \frac{y_2(x) y_1(\xi)}{w(\xi)} = \frac{(e^\xi - e^{-\xi})(e^x - e^{2-x})}{(2e^2 - 2)} & \xi < x \leq 1 \end{cases}$$

It follows that the solution to the boundary value problem is given by

$$u(x) = \int_0^1 G(x, \xi)(\xi^4 + 1) d\xi.$$

Problem 5 (4 points)

For which $k \in \mathbb{R}$ and $L > 0$ does there exist a non-trivial solution on the interval $[0, L]$ to the equation $y''(t) + ky(t) = 0$ with $y(0) = y(L) = 0$? Prove your answer.

Solution: The equation $y''(t) + ky(t) = 0$ is linear with constant coefficients, hence the Picard-Lindelöf theorem implies that the space of solutions is two-dimensional. We have to check for which k and L it contains a non-trivial function satisfying the boundary conditions.

Case 1, $k = 0$: In that case the solution space consists of all (affine) linear functions $at + b$. Every non-trivial such function only intersects the 0-line in at most one point. Hence there does not exist a non-trivial solution.

Case 2, $k < 0$: A general solution is of the form $x(t) = ae^{\sqrt{-k}t} + be^{-\sqrt{-k}t}$. The requirement $x(0) = 0$ forces $a = -b$. Hence the second boundary condition becomes

$$a(e^{\sqrt{-k}L} - e^{-\sqrt{-k}L}) = 0.$$

Note that $e^{\sqrt{-k}L} > 1$ for all $L > 0$, and $e^{-\sqrt{-k}L} < 1$ for all $L > 0$. Hence there cannot be a non-trivial solution.

Case 3, $k > 0$: A general solution is of the form $x(t) = a\sin(\sqrt{k}t) + b\cos(\sqrt{k}t)$. The requirement $x(0) = 0$ forces $b = 0$. The second boundary condition than requires $x(L) = a\sin(\sqrt{k}L) = 0$. For $a \neq 0$ this holds if and only if $\sqrt{k}L$ is an integer-multiple of π, i.e., when $\sqrt{k}L = n\pi$ for some $n \in \mathbb{Z}$.

Problem 6 (4 points)

(1) Show that the autonomous systems

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

and

$$\begin{cases} x' = y(x^2 + y^2) \\ y' = -x(x^2 + y^2) \end{cases}$$

have the same orbits. That is, their solution curves only differ by a reparametrization. (Hint: First describe the solution curves for the first autonomous system and then show that a reparametrization solves the second).

(2) Compute the equilibrium points for both systems and determine whether they are stable, asymptotically stable or unstable.

Solution: (1): The solution curves to the first system are given by $(r\sin(t), r\cos(t))$ for r a non-negative real number, and shifts thereof. These describe circles of radius r around the origin and hence cover all the points.
For the second system we note that the expression $x^2 + y^2$ is constant on each of these circles, i.e., the orbits of the first system. Hence, the phase portrait of the second system is the same as that of the first, only scaled by the square of the radius of the respective circle. It follows that the solution curve to the second system is a rescaling of that of the first by that factor. Hence we find that $(r\sin(r^2t), r\cos(r^2t))$ describes the solution curves to the second (which a quick check verifies).

(2): Equilibrium points as well as their stability only depend on the orbits, which in both cases we have computed to be the circles around the origin. It follows that 0 is the only equilibrium point, which is stable but not asymptotically stable.