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Suggested solutions

Exam: Introduction to Finance Mathematics (MT5009), 2023-08-25

Problem 1

See Capinski & Zastawniak, for example ch. 1, 4, 5.

Problem 2

(A) The weights of the market portfolio is (see Capinski & Zastawniak p. 83)

$$\mathbf{w}_M = \frac{(\mathbf{m} - R\mathbf{u})\mathbf{C}^{-1}}{(\mathbf{m} - R\mathbf{u})\mathbf{C}^{-1}\mathbf{u}^T}$$

where $\mathbf{u}$ is a vector of ones, and in our case

$$\mathbf{m} = (0.05, 0.06)$$

and

$$\mathbf{C} = \begin{pmatrix} 0.1^2 & 0.1 \cdot 0.12 \cdot 0.2 \\ 0.1 \cdot 0.12 \cdot 0.2 & 0.12^2 \end{pmatrix}$$

i.e.,

$$\mathbf{C} = \begin{pmatrix} 0.01 & 0.0024 \\ 0.0024 & 0.0144 \end{pmatrix}.$$

Standard calculations give

$$\mathbf{C}^{-1} = \begin{pmatrix} 104,1667 & -17,3611 \\ -17,3611 & 72,3380 \end{pmatrix}.$$

Lastly, we use the formula above and find with basic calculations

$$\mathbf{w}_M = (0.5060, 0.4940).$$

(B) The solution to this variance minimization problem is given in Capinski & Zastawniak p. 73, as $\mathbf{w}_{MVP} = \frac{\mathbf{u}\mathbf{C}^{-1}}{\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^T}$. Relying on the information in **(A)** above we find with some calculations that

$$\mathbf{w}_{MVP} = (0.6122, 0.3878)\,.$$

Problem 3

(A) Write $S^u = S(0)(1+U) = 22$ and $S^u = S(0)(1+D) = 19$. The replicating portfolio is given by

$$\begin{aligned} x(1)S^u + y(1)(1+R) &= max\{S^u - X; 0\} \\ x(1)S^d + y(1)(1+R) &= max\{S^d - X; 0\}, \end{aligned}$$

where $x(1)$ is the number of shares and $y(1)$ is the amount of money in the risk-free asset in the replicating portfolio. By plugging in numbers we see that the replicating portfolio should satisfy

$$\begin{aligned} x(1)22 + y(1) &= 4 \\ x(1)19 + y(1) &= 1. \end{aligned}$$

Solving this equation system gives the replicating portfolio $x(1) = 1$ and $y(1) = -18$. The cost of the replicating portfolio is $x(1)S(0) + y(1) = 1 \cdot 20 - 18 = 2$.

(B) Replicating the call means buying shares at $t = 0$ and selling them at $t = 1$ (in this case incurring a proportional cost of 5%). Hence, the replicating portfolio should in this case solve

$$\begin{aligned} x(1)(1-0.05)S^u + y(1)(1+R) &= max\{S^u - X; 0\} \\ x(1)(1-0.05)S^d + y(1)(1+R) &= max\{S^d - X; 0\}. \end{aligned}$$

By plugging in numbers we find that this boils down to

$$\begin{aligned} x(1)20.9000 + y(1) &= 4 \\ x(1)18.0500 + y(1) &= 1. \end{aligned}$$

Solving this equation system gives the replicating portfolio

$$\begin{aligned} x(1) &= 1.0526, \\ y(1) &= -18. \end{aligned}$$

The cost of the replicating portfolio is

$$x(1)S(0) + y(1) = 1.0526 \cdot 20 - 18 = 3.0526.$$

Problem 4

This is based on Capinski & Zastawniak around p. 227. We aim at finding the value VaR that solves the equation

$$P(10S(0)e^r - 10S(1) < VaR) = 0.95$$

Use that the random variables $S(1)$ and $S(0)e^{\mu+\sigma Z}$ (where $Z \sim N(0,1)$) have the same distribution, to see that the above equation is equivalent to

$$P(10S(1) > 10S(0)e^r - VaR) = 0.95$$

$$\Leftrightarrow P\left(e^{\mu+\sigma Z} > e^r - \frac{VaR}{10S(0)}\right) = 0.95$$

$$\Leftrightarrow P\left(\mu + \sigma Z > \ln\left(e^r - \frac{VaR}{10S(0)}\right)\right) = 0.95$$

$$\Leftrightarrow P\left(Z > \frac{\ln\left(e^r - \frac{VaR}{10S(0)}\right) - \mu}{\sigma}\right) = 0.95$$

$$\Leftrightarrow 1 - P\left(Z < \frac{\ln\left(e^r - \frac{VaR}{10S(0)}\right) - \mu}{\sigma}\right) = 0.95$$

$$\Leftrightarrow P\left(Z < \frac{\ln\left(e^r - \frac{VaR}{10S(0)}\right) - \mu}{\sigma}\right) = 0.05$$

Recalling the value for standard normal quantile $N^{-1}(0.05) \approx -1.645$, this yields

$$\frac{\ln\left(e^r - \frac{VaR}{10S(0)}\right) - \mu}{\sigma} = -1.645.$$

Plugging in values for $S(0)$, r, μ and σ and solving for VaR yields

$$VaR = 25.59.$$

Problem 5

(A) We find

$$\begin{aligned}
p_* &= \frac{R - D}{U - D} = 1/3, \\
S^{uu} &= S(0)(1+U)^2 = 24.20, \\
S^{ud} &= S(0)(1+U)(1+D) = 20.90, \\
S^{dd} &= S(0)(1+D)^2 = 18.05.
\end{aligned}$$

The risk-neutral valuation formula for the put is

$$\begin{aligned}
&P_E(0) \\
&= \frac{1}{(1+R)^2} E_*\left[(X - S(2))_+\right] \\
&= \frac{1}{(1+R)^2}\left[p_*^2(X - S^{uu})_+ + 2p_*(1-p_*)(X - S^{ud})_+ + (1-p_*)^2(X - S^{dd})_+\right].
\end{aligned} \tag{1}$$

Plugging in the numbers above and basic calculations give

$$P_E(0) = 0.8667.$$

(B) Similar to the above we find (with notation in line with Capinski & Zastawniak) that

$$P_E^u = \frac{1}{(1+R)} \left[p_*(X - S^{uu})_+ + (1-p_*)(X - S^{ud})_+ \right]$$
$$= 0$$

and

$$P_E^d = \frac{1}{(1+R)} \left[p_*(X - S^{ud})_+ + (1-p_*)(X - S^{dd})_+ \right]$$
$$= 1.300.$$

Using this we can describe the replicating strategy for the option as follows:

The replicating portfolio formed at time 0 is given by

$$x(1) = \frac{P_E^u - P_E^d}{S^u - S^d} = -0.4333$$
$$y(1) = \frac{P_E^u - x(1)S^u}{1+R} = 9.5333.$$

The replicating portfolio formed at time 1 in case $S(1) = S^u$ is given by

$$x^u(2) = \frac{P_E^{uu} - P_E^{ud}}{S^{uu} - S^{ud}} = 0$$
$$y^u(2) = \frac{P_E^{uu} - x^u(2)S^{uu}}{1+R} = 0.$$

The replicating portfolio formed at time 1 in case $S(1) = S^d$ is given by

$$x^d(2) = \frac{P_E^{ud} - P_E^{dd}}{S^{ud} - S^{dd}} = -0.6842,$$
$$y^d(2) = \frac{P_E^{ud} - x^d(2)S^{ud}}{1+R} = 14.3000.$$

(C) If we view the option value $P_E(0)$ in Equation (1) in **(A)** above as a function of the strike $X > 0$ it is clear that it is an increasing function such that $X \to \infty \Rightarrow P_E(0) \to \infty$, and $X \to 0 \Rightarrow P_E(0) \to 0$, and that there exists exactly one strike price X such that $P_E(0) = 10$.