MT7047 – Probability theory III – exam

Date Monday October 23, 2023
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Tools None.
Grading criteria The exam consists of two parts, which consist of 20 and 40 points respectively. To pass the exam a score of 14 or higher is required on Part I. If attained, then also Part II is graded, and the score on this part determines the grade. Grades are determined according to the following table:

	A	B	C	D	E
Part I	14	14	14	14	14
Part II	32	24	16	8	0

Problems of Part I may give up to five points each, and problems of Part II may give up to ten points each. Complete and well motivated solutions are required for full score. Partial solution may be rewarded with a partial score.

Part I

Problem 1. Let $\mathcal{F}$ and $\mathcal{G}$ be σ-algebras on some set Ω. Show that $\mathcal{F} \cap \mathcal{G}$ are again a σ-algebra on Ω.

Problem 2. Roll a six-sided die three times. Let X denote the outcome of the first roll, and Y the sum of the three outcomes.

(a) Construct a probability space corresponding to the above experiment.

(b) Explain why X and Y are random variables.

(c) Determine the conditional expectation $\mathbb{E}[Y|X]$.

Problem 3. An urn contains one red and one blue ball initially. In each round a ball is drawn uniformly at random from the urn, and then replaced along with an additional blue ball. Note that a blue ball is added irregardless of the colour of the ball drawn. Compute the probability that the red ball is drawn infinitely many times.

Problem 4. Let $X_1, X_2, \dots$ be independent random variables where $X_k = \pm 2^{-k}$ with equal probability, and set $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Show that $S_n = \sum_{k=1}^{n} X_k$ defines a martingale with respect to $\mathcal{F}_n$. Moreover, show that the limit $\lim_{n\to\infty} S_n$ exists almost surely, and compute its expectation.

Part II

Problem 5. Construct a probability space corresponding to the experiment of rolling a six-sided die until the die comes up 'six'. Let N denote the number of rolls required, and X the number of times the die comes up 'three'. Determine the distribution of N, its mean, and compute $\mathbb{E}[X]$.

Hint: Recall that $\sum_{k\geq 1} kp(1-p)^{k-1} = 1/p$.

Problem 6. Let X be a random variable on some probability space whose distribution function is given by

$$F_X(x) = \begin{cases} x/3 & \text{for } x \in [0,1), \\ (x+1)/3 & \text{for } x \in [1,2]. \end{cases}$$

In addition, let U and Z be independent random variables, where U is uniformly distributed on $[0,2]$ and Z is Bernoulli distributed with parameter $1/3$, and define $Y = Z + U(1-Z)$. Show that X and Y are equal in distribution, and compute their expectation.

Problem 7. Let $X_1, X_2, \ldots$ be independent random variables taking values ± 1 with equal probability. Let $S_n = X_1 + \ldots + X_n$, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and set $S_0 = 0$.

(a) Show that $(S_n)_{n\geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$.

(b) Let $T = \min\{n \geq k : S_n = S_{n-k} + k\}$ and $U = \min\{n \geq 0 : S_{n+k} = S_n + k\}$, and determine whether T and U are stopping times with respect to $(\mathcal{F}_n)_{n\geq 1}$ or not.

(c) Compute $\mathbb{E}[S_T]$ and $\mathbb{E}[S_U]$.

Problem 8. Let $X_1, X_2, \ldots$ be independent random variables that are uniformly distributed on the interval $[1/2, 3/2]$. Let $Y_n = \prod_{k=1}^{n} X_k$.

(a) Show that, almost surely, $\frac{1}{n}\log Y_n \to c < 0$ as $n \to \infty$.

(b) Show that, almost surely, $Y_n \to 0$ as $n \to \infty$.