Probability theory III : Solutions to exam October 23, 2023

Part I

1 Let $\mathcal{F}$ and $\mathcal{G}$ be σ-algebras on Ω. Then:

(i) $\Omega \in \mathcal{F}$ and $\Omega \in \mathcal{G}$, so $\Omega \in \mathcal{F} \cap \mathcal{G}$.

(ii) If $A \in \mathcal{F} \cap \mathcal{G}$, then $A^c \in \mathcal{F}$ and $A^c \in \mathcal{G}$, so $A^c \in \mathcal{F} \cap \mathcal{G}$.

(iii) Suppose $A_1, A_2, \ldots$ are all in $\mathcal{F} \cap \mathcal{G}$. Then their union is in both $\mathcal{F}$ and $\mathcal{G}$, and hence in the intersection. 5p

Thus, $\mathcal{F} \cap \mathcal{G}$ satisfies all properties of a σ-algebra.

2 (a) We may take

$$\Omega = \{(a,b,c) : a,b,c = 1,2,\ldots,6\}$$
$$\mathcal{F} = \text{power set of } \Omega$$
$$\mathbb{P}((a,b,c)) = (1/6)^3 \quad \text{for every } (a,b,c) \in \Omega.$$ 2p

(b) Since $\mathcal{F}$ is the power set, any function $X : \Omega \to \mathbb{R}$ is a random variable. 1p

(c) Let X_k denote the outcome of roll k. Then $Y = X_1 + X_2 + X_3$.

$$\mathbb{E}[Y \mid X] = \mathbb{E}[X_1 + X_2 + X_3 \mid X_1] \overset{\text{linearity}}{=} \mathbb{E}[X_1 \mid X_1] + \mathbb{E}[X_2 \mid X_1] + \mathbb{E}[X_3 \mid X_1]$$
$$\underset{\text{take out known}}{\overset{\text{indep}}{=}} X_1 + \mathbb{E}[X_2] + \mathbb{E}[X_3] = X + 7.$$ 2p

3 Let A_k denote the event that a red ball is drawn in round k. After $k-1$ rounds there are $k+1$ balls in the urn, and since each ball is equally likely to be chosen,

$$\mathbb{P}(A_k) = \frac{1}{k+1}.$$ 2p

Since draws can be assumed to be independent, and

$$\sum_{k=1}^{\infty} \mathbb{P}(A_k) = \sum_{k=1}^{\infty} \frac{1}{k+1} = \infty,$$ 3p

it follows by the second Borel-Cantelli lemma that A_k will occur for infinitely many k with probability one.

$\boxed{4}$ Clearly S_n is $\mathcal{F}_n$-measurable, since a function of $X_1, \dots, X_n$. Moreover,

$$\mathbb{E}[|S_n|] \le \sum_{k=1}^{n} \mathbb{E}[|X_k|] \le \sum_{k\ge 1} 2^{-k} = 1,$$

and

$$\mathbb{E}[S_{n+1} | \mathcal{F}_n] = S_n + \mathbb{E}[X_{n+1} | \mathcal{F}_n] = S_n.$$ 2p

Hence $(S_n)_{n\ge 1}$ is a bounded martingale, and hence almost surely convergent. By the bounded convergence theorem we conclude that 3p

$$\mathbb{E}[\lim_{n\to\infty} S_n] = \lim_{n\to\infty} \mathbb{E}[S_n] = \mathbb{E}[S_0] = 0.$$

Part II

$\boxed{5}$ We may take the following:

$$\Omega = \bigcup_{n=1}^{\infty} \{(\omega_1, \omega_2, \dots, \omega_{n-1}, 6) : \omega_i \in \{1, 2, \dots, 5\} \text{ for } i = 1, \dots, n-1\}$$

$\mathcal{F}$ = power set of Ω (ok, since Ω countable)

$\mathbb{P}((\omega_1, \dots, \omega_{n-1}, 6)) = \left(\frac{1}{6}\right)^n$ for every $\omega \in \Omega$ of length n. 4p

Note that the event $\{N=k\}$ contains all sequences in Ω of length k, which are 5^{k-1}. Hence

$$\mathbb{P}(N=k) = 5^{k-1} \cdot \frac{1}{6^k} = \left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{6}, \quad \text{for } k \ge 1.$$

It follows that 3p

$$\mathbb{E}[N] = \sum_{k=1}^{\infty} k \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{k-1} = 6.$$

Finally we compute the expectation of X, by conditioning on N. Note that conditional on $N=k$, X is $\mathrm{Bin}(k-1, 1/5)$-distr., since there are $\binom{k-1}{\ell} 4^{k-1-\ell}$ sequences length k that contains ℓ threes, and ends with six, so

$$\mathbb{P}(X=\ell | N=k) = \frac{\mathbb{P}(X=\ell, N=k)}{\mathbb{P}(N=k)} = \frac{\binom{k}{\ell} \cdot 4^{k-\ell} \cdot \frac{1}{6^k}}{(5/6)^{k-1}\ 1/6}$$

$$= \binom{k-1}{\ell} \left(\frac{1}{5}\right)^{\ell} \left(\frac{4}{5}\right)^{k-1-\ell}$$

Hence, by properties of conditional expectation

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|N]] = \mathbb{E}[(N-1)\cdot\tfrac{1}{5}] = \frac{\mathbb{E}[N]-1}{5} = 1.$$

3p

$\boxed{6}$ Let $Y = Z + U(1-Z)$ where $Z \sim \mathrm{Bern}(1/3)$ and $U \sim \mathrm{unif}[0,2]$ are independent. We determine the distribution function of Y to show that it coinsides with F_X. For $y \in [0,1)$

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(Y \le y | Z=0)\mathbb{P}(Z=0) + \mathbb{P}(Y \le y | Z=1)\mathbb{P}(Z=1)$$
$$= \mathbb{P}(U \le y)\cdot\mathbb{P}(Z=0) = \frac{y}{2}\cdot\frac{2}{3} = \frac{y}{3}$$

For $y \in [1,2]$

5p

$$F_Y(y) = \mathbb{P}(Y \le y | Z=0)\mathbb{P}(Z=0) + \mathbb{P}(Y \le y | Z=1)\mathbb{P}(Z=1)$$
$$= \mathbb{P}(U \le y)\mathbb{P}(Z=0) + \mathbb{P}(1 \le y)\cdot\mathbb{P}(Z=1)$$
$$= \frac{y}{2}\cdot\frac{2}{3} + 1\cdot\frac{1}{3} = \frac{y}{3} + \frac{1}{3}.$$

For $y \le 0$ we have $F_Y(y) = 0$ and for $y \ge 2$ we have $F_Y(y) = 1$. Hence X and Y are equal in distribution. Hence their expectations are equal.

$$\mathbb{E}[X] = \mathbb{E}[Y] \overset{\text{linearity}}{=} \mathbb{E}[Z] + \mathbb{E}[U(1-Z)]$$
$$\overset{\text{indep. + lin.}}{=} \mathbb{E}[Z] + \mathbb{E}[U](1 - \mathbb{E}[Z])$$
$$= 1/3 + 1\cdot(1 - 1/3)$$
$$= 1.$$

5p

$\boxed{7}$ (a) Clearly S_n is $\mathcal{F}_n$-measurable, and $\mathbb{E}[|S_n|] \le n$. Moreover,

$$\mathbb{E}[S_{n+1} | \mathcal{F}_n] = S_n + \mathbb{E}[X_{n+1}] = S_n.$$

Hence $(S_n)_{n \ge 0}$ is a martingale.

2p

(b) We note that T denotes the first time n that the RW $(S_n)_{n\geq 0}$ completes k consecutive steps upwards for the first time. That is, the least n so that $X_1, X_2, \ldots, X_n$ contains k 1s in a row. Hence $\{T=n\}$ is clearly $\mathcal{F}_n$-measurable, and T a stopping time wrt $(\mathcal{F}_n)_{n\geq 1}$.

However, U is <u>not</u> a stopping time wrt $(\mathcal{F}_n)_{n\geq 1}$, since in order to determine $\{U=n\}$ we need to know $X_1, X_2, \ldots, X_{n+k}$.

3p

(c) We investigate the conditions of Optional stopping III. First $|S_{n+1}-S_n| \leq 1$, so it will suffice to show $\mathbb{E}[T]<\infty$.

Let $A_n = \{X_{n+1} = \ldots = X_{n+k} = 1\}$. Then

$$\mathbb{P}(A_n) = 1/2^k$$

$$\mathbb{P}(T>n) = \mathbb{P}\Big(\bigcap_{\ell=1}^{n-k} A_\ell^c\Big)$$

$$\leq \mathbb{P}\Big(\bigcap_{j=1}^{\lfloor n/k \rfloor} A^c_{(j-1)k}\Big)$$

event regard disjoint parts of sequence

$$= \mathbb{P}(A_0^c)^{\lfloor \frac{n}{k} \rfloor}$$

$$\leq \underbrace{\Big[(1-2^{-k})^{1/k}\Big]}_{c<1}{}^{n-1}$$

3p

It follows that

$$\mathbb{E}[T] = \sum_{n\geq 0} \mathbb{P}(T>n) \leq \sum_{n\geq 0} c^{n-1} < \infty.$$

Optional stopping III hence gives

$$\mathbb{E}[S_T] = \mathbb{E}[S_0] = 0.$$

Since $U = T-k$ and $S_U = S_{T-k} = S_T - k$ we have

$$\mathbb{E}[S_U] = \mathbb{E}[S_T] - k = -k.$$

2p

$\boxed{8}$ Let $X_1, X_2, \ldots$ be independent unif $[\frac{1}{2}, \frac{3}{2}]$.

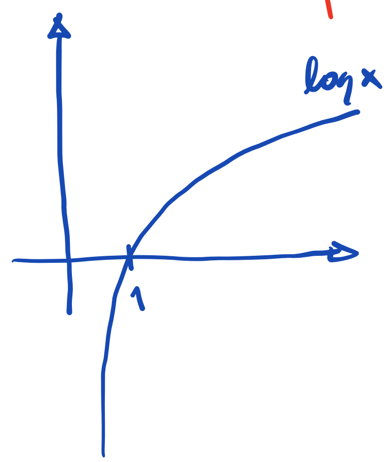
$$Y_n = \prod_{k=1}^{n} X_k .$$

(a) By the law of large numbers we have, almost surely,

$$\frac{1}{n} \log Y_n = \frac{1}{n} \sum_{k=1}^{n} \log X_k \longrightarrow \mathbb{E}[\log X_1]$$

as $n \to \infty$.

3p



We have

$$\mathbb{E}[\log X_1] = \int_{\frac{1}{2}}^{3/2} \log x \, dx$$

$$= \int_{\frac{1}{2}}^{1} \log x \, dx + \int_{1}^{3/2} \log x \, dx .$$

Since the derivative of $\log x$ is strictly decreasing, it follows that

$$-\int_{\frac{1}{2}}^{1} \log x \, dx > \int_{1}^{3/2} \log x \, dx$$

and hence that

$$\mathbb{E}[\log X_1] < 0 .$$

3p

(Alternatively, compute integral.)

(b) From (a) we have that, almost surely, for all large n we have

$$\log Y_n \leq \frac{c}{2} n ,$$

which is equivalent to

$$Y_n \leq e^{cn/2} .$$

4p

Since $c < 0$ the RHS $\to 0$ as $n \to \infty$, and hence with probability one we have $Y_n \to 0$ as $n \to \infty$.