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Theory of Statistical Inference

Exam, 2023/10/25

The only allowed aid is a pocket calculator provided by the department. The answers to the tasks should be clearly formulated and structured. The answers should be simplified as far as possible. All non-trivial steps need to be commented. The solutions should be given in English or Swedish.

The written exam is divided into two parts. The first part considers the most central of the course concepts and it is related to standard problems. The second part consists of problems that requires a higher level of understanding, the ability to generalize and to combine methods. Each part consists of three problems and will worth a maximum of 50 points. In order to receive grades A-E, a minimum of 35 points is required in the first part. The second part is only graded for students passing the first part. Given a minimum of 35 points in the first part, the final grade is determined by the sum of regular points in both parts of the exam and bonus points according to the following table:

Grade	A	B	C	D	E	F
Points	$\geq$90	(90-80]	(79-70]	(69-60]	< 60 and ≥ 35 in Part I	< 35 in Part I

Up to 10 bonus points (i.e., in addition to the ordinary 100 points) are given for the active participation in the problem sessions. A half of the bonus points will be used for the first part of the exam, while the second half of the bonus points will be used in the second part of the exam.

Part I:

Problem 1 [20P]

Let $X_{1:n} = (X_1, X_2, ..., X_n)$ be an iid sample from a gamma distribution with density of X_i given by

$$f_{X_i}(x;\theta) = \frac{\theta^\beta}{\Gamma(\beta)} x^{\beta-1} \exp(-\theta x) \qquad \text{for} \qquad x > 0, \quad \theta > 0,$$

where $\beta > 0$ is assumed to be known and $\Gamma(.)$ denotes the gamma function.

(a) Derive the maximum likelihood estimate $\hat{\theta}_{ML}$ for θ. **[5P]**

(b) Derive the ordinary Fisher information $I_{1:n}(\theta)$, the observed Fisher information $I_{1:n}(\hat{\theta}_{ML})$, and the expected Fisher information $J_{1:n}(\theta)$. **[4P]**

(c) Derive the test statistic of the score test for the null hypothesis $H_0 : \theta = 1$. Simplified the expression of the test statistic as much as possible. **[4P]**

(d) Derive the test statistic of the Wald test for the null hypothesis $H_0 : \theta = 1$. Simplified the expression of the test statistic as much as possible. **[4P]**

(e) Perform the score test and the Wald test at the significance level of 5%, when $\beta = 4$, $n = 25$ and $\sum_{i=1}^{n} x_i = 81.7$? **[3P]**

Hint: *Important quantiles of the standard normal distribution are:*

$z_{0.9}$	$z_{0.95}$	$z_{0.975}$	$z_{0.995}$
1.28	1.64	1.96	2.33

Problem 2 [20P]

Let $X_{1:n} = (X_1, ..., X_n)$ be an iid sample from a normal distribution with density of X_i, $i = 1, ..., n$, given by

$$f_{X_i}(x;\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \qquad \text{for} \quad x \in \mathbb{R},\ \mu \in \mathbb{R},$$

where $\sigma^2 > 0$ is assumed to be known.

(a) Show that the conjugate prior for μ is given by the normal distribution with parameters $\mu_0 \in \mathbb{R}$ and $\sigma_0^2 > 0$ and specify the parameters of the corresponding posterior distribution. **[5P]**

(b) Compute two Bayesian estimators for μ when the conjugate prior is used. **[3P]**

(c) Derive the expression of a 95% credible interval for μ when the conjugate prior is used. **[2P]**

(d) Find the expression of the Jeffreys prior for μ and compute the corresponding posterior distribution. **[5P]**

(e) Compute two Bayesian estimators and a 95% credible interval for μ when the Jeffreys prior is used. **[5P]**

Problem 3 [10P]

Provide the expression of the likelihood ratio (LR) statistic in the case of a scalar parameter. What is the asymptotic distribution of the LR statistic. Describe how a likelihood ratio confidence interval is constructed.

Part II:

Problem 4 [17P]

Let $X_{1:n_1} = (X_1, ..., X_{n_1})$ be an iid sample from a normal distribution with density of X_i, $i = 1, ..., n_1$, given by

$$f_{X_i}(x; \mu_1, \sigma_1^2) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right) \quad \text{for} \quad x \in \mathbb{R}, \mu_1 \in \mathbb{R}, \sigma_1^2 > 0$$

and let $X_{n_1+1:n_1+n_2} = (X_{n_1+1}, ..., X_{n_1+n_2})$ be an iid sample from a normal distribution with density of X_j, $j = n_1+1, ..., n_1+n_2$, given by

$$f_{X_j}(x; \mu_2, \sigma_2^2) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(x-\mu_2)^2}{2\sigma_2^2}\right) \quad \text{for} \quad x \in \mathbb{R}, \mu_2 \in \mathbb{R}, \sigma_2^2 > 0$$

Assume that $X_{1:n_1}$ and $X_{n_1+1:n_1+n_2}$ are independent.

The aim is to test the null hypothesis:

$$H_0: \sigma_1^2 = \sigma_2^2. \tag{1}$$

(a) Derive the generalized likelihood ratio statistic for testing H_0 in (1). Simplified the expression of the test statistic as much as possible. **[12P]**

(b) Determine the distribution of the test statistics derived in part (a). **[1P]**

(c) Perform the generalized likelihood ratio test at significance level of 5% when $n_1 = 15$, $n_2 = 25$, $\sum_{i=1}^{n_1} x_i = 12$, $\sum_{j=n_1+1}^{n_1+n_2} x_j = 33$, $\sum_{i=1}^{n_1}(x_i - \bar{x}_1)^2 = 14$ and $\sum_{j=n_1+1}^{n_1+n_2}(x_j - \bar{x}_2)^2 = 51$ where $\bar{x}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i$ and $\bar{x}_2 = \frac{1}{n_2}\sum_{j=n_1+1}^{n_1+n_2} x_j$. **[4P]**

Hint: *Important quantiles of the χ^2-distribution at various degrees of freedom are:*

d	1	2	3	4	5
$\chi^2_{0.9}(\text{df} = d)$	2.71	4.61	6.25	7.78	9.24
$\chi^2_{0.95}(\text{df} = d)$	3.84	5.99	7.81	9.49	11.07
$\chi^2_{0.975}(\text{df} = d)$	5.02	7.38	9.35	11.14	12.83

Problem 5 [18P]

Let $X_{1:n} = (X_1, ..., X_n)$ be an iid sample from a Poisson distribution with probability mass function of X_i, $i = 1, ..., n$, given by

$$\mathbb{P}(X_i = x; \lambda) = \frac{\lambda^x}{x!} \exp(-\lambda) \quad \text{for} \quad \lambda > 0 \quad \text{and} \quad x = 0, 1, 2, ...$$

Furthermore, it holds that $\mathbb{E}(X_i) = \lambda$.

(a) Derive the maximum likelihood estimate $\hat{\lambda}_{ML}$ for λ. **[3P]**

(b) Derive the ordinary Fisher information $I_{1:n}(\lambda)$ and the expected Fisher information $J_{1:n}(\lambda)$. **[3P]**

(c) Determine the asymptotic distribution of $\hat{\lambda}_{ML}$. Simplified the expression of the asymptotic distribution as much as possible.**[2P]**

(d) Specify the variance stabilising transformation $\phi = h(\lambda)$. **[3P]**

(e) Derive the maximum likelihood estimate for ϕ and determine its asymptotic distribution. Simplified the expression of the asymptotic distribution as much as possible. **[3P]**

(f) Using the result of part (e), find a 95% confidence interval for λ.**[4P]**

Problem 6 [15P]

Formulate the statement of the Cramér-Rao lower bound and prove it.