Solutions Exam 23/10/2023

① $f(0)=0$ 0.5

$f'(x) = \ln(1-x^2) + x\frac{1}{1-x^2}(-2x) = \ln(1-x^2) - \frac{2x^2}{1-x^2}$ 0.5

$f'(0) = \ln(1) + 0 = 0$ 0.5

$f''(x) = \frac{1}{1-x^2}(-2x) - \frac{4x(1-x^2)+2x^2\cdot 2x}{(1-x^2)^2}$ 0.5

$= \frac{-1}{1-x^2}\cdot 2x - \frac{4x - \cancel{4x^3} + \cancel{4x^3}}{(1-x^2)^2}$

$= \frac{-1}{1-x^2}\cdot 2x - \frac{4x}{(1-x^2)^2}$

$f''(0) = 0$ 0.5

$f'''(x) = \frac{2(1-x^2)+2x(2x)}{(1-x^2)^2} - \frac{4(1-x^2)^2 + 4x\,2x\cdot 2(1-x^2)}{(1-x^2)^4}$

$= \frac{2(1-x^2)^3 + 4x^2(1-x^2)^2 - 4(1-x^2)^2 + 16x^2(1-x^2)}{(1-x^2)^4}$

$f'''(0) = -2 - 4 = -6$ 0.5

0.5

$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{6}(x-0)^3$

$= -x^3$ 0.5

$P_3(0.1) = \frac{-1}{1000}$ 0.5

(2) 10 years = 120 months

6% a year = 0.5% a month

The Debt is 500000 − 200000 = 300000

After the first ~~year~~ month   0.5 for setting up

The debt is $300000\left(1+\frac{1}{200}\right) - m$

After the second payment this is

(a) $\boxed{300000\left(1+\frac{1}{200}\right)^2 - m\left(1+\frac{1}{200}\right) - m}$ 0.5

answer

(b) we have that

$$0 = 300000\left(1+\frac{1}{200}\right)^{120} - \sum_{i=0}^{119} m\left(1+\frac{1}{200}\right)^i$$

$$= 300000\left(1+\frac{1}{200}\right)^{120} - m\,\frac{\left(1+\frac{1}{200}\right)^{120}-1}{\frac{1}{200}}$$

0.5

$$= 300000\left(1+\frac{1}{200}\right)^{120} - m\cdot 200\left(\left(1+\frac{1}{200}\right)^{120}-1\right)$$

We solve for m

$$m = \frac{300000\left(1+\frac{1}{200}\right)^{120}}{200\left(\left(1+\frac{1}{200}\right)^{120}-1\right)}$$

$$\approx 1500\,\frac{1.82}{.82} = \boxed{3328.16}$$ 0.5

(c) $700000 = $ capital

monthly interest $= \frac{1}{3}\%$

After a month I have

$$700000\left(1+\frac{1}{300}\right) - m$$

After two months

$$700000\left(1+\frac{1}{300}\right)^2 - m\left(1+\frac{1}{300}\right) - m$$

After 3 month

$$700000\left(1+\frac{1}{300}\right)^3 - \sum_{i=0}^{2} m\left(1+\frac{1}{300}\right)^i =$$

$$= 700000\left(1+\frac{1}{300}\right)^3 - 3328.16\left(\frac{\left(1+\frac{1}{300}\right)^3 - 1}{\frac{1}{300}}\right)$$

$$\approx 700000\underbrace{\left(1+\frac{1}{300}\right)^3}_{\simeq 1.01} - 998448\,(0.01)$$

$\simeq$ ~~687016~~ 697015.52   0.5

After one year

$$700000\left(1+\frac{1}{300}\right)^{12} - \sum_{i=0}^{11} m\left(1+\frac{1}{300}\right)^i$$

$$= 700000\left(1+\frac{1}{300}\right)^{12} - 3328.16\left(\frac{\left(1+\frac{1}{300}\right)^{12} - 1}{\frac{1}{300}}\right)$$

$$= 700000\left(1+\frac{1}{300}\right) - 998448\left(\left(1+\frac{1}{300}\right)^{12} - 1\right)$$

$\approx 728519.08 - 40678.31 \approx 687840.7$ 0.5

(d) Paying cash after 10 years we have

$$200\,000\left(1+\frac{1}{300}\right)^{120} = \cancel{298166.54}\ \cancel{2740800000}$$

$$298166.54$$

~~After~~ with the payment plan we hav

$$700000\left(1+\frac{1}{300}\right)^{120} - \sum_{i=0}^{119} m\left(1+\frac{1}{300}\right)^{i}$$

$$= 700\,000\left(1+\frac{1}{300}\right)^{120} - m\left(\frac{\left(1+1/300\right)^{120}-1}{\frac{1}{300}}\right)$$

$$= 700000\left(1+\frac{1}{300}\right)^{120} - 998448\left(\left(1+\frac{1}{300}\right)^{120}-1\right)$$

$$\approx 10\cancel{43582.88} \to \cancel{998400}\ 553581.97$$

Thus paying with the payment plan is more advantagious

(3)

(a) $f'(x) = \dfrac{2x(x+1)-(x^2+8)}{(x+1)^2} = \dfrac{2x^2+2x-x^2-8}{(x+1)^2}$

$= \dfrac{x^2+2x-8}{(x+1)^2}$ 0.5

we have that

$f'(x)=0$ iff $x^2+2x-8=0$

$\iff x_\pm = -1 \pm \sqrt{1+8} = -1 \pm 3$

$x=2$

$x=-4$ 0.5

we study the signe of $f'(x)$

		-4		2	
$f'(x)$	+	0	−	0	+
$f(x)$	↗		↘		↗
		MAX		MIN	

0.5

<u>Answer</u> there are two critical point $x=-4$ is a local max $x=2$ is a local min 0.5

(b) From (a) we have that the function is increasing when $x \in (-\infty,-4) \cup (2,+\infty)$ and decreasing otherwise 1.

For the convexity we have to compute the second derivative.

$$f''(x) = \frac{(2x+2)(x+1)^2 - 2(x+1)(x^2+2x-8)}{(x+1)^4} =$$

$$= \frac{(2x+2)(x^2+2x+1) - (2x+2)(x^2+2x-8)}{(x+1)^4}$$

$$= \frac{(2x+2)(\cancel{x^2}+\cancel{2x}+1 - \cancel{x^2}-\cancel{2x}+8)}{(x+1)^4} =$$

$$= \frac{18}{(x+1)^3}$$

This is positive when $x > 1$

~~negative other~~

$+$

The function is convex $x \geq 1$

concave when $x < -1$

(c) we have one critical point in the interval

$f(1) = \frac{9}{2}$ maximum

$f(2) = \frac{12}{3} = 4$ minimum

$f(3) = \frac{17}{4}$

The maximum value of the function is $\frac{9}{2}$ taken at $x = 1$ and the minimum value is 4 taken at $x = 2$.

(4)

(a) $\int \frac{3}{\sqrt{t}} e^{\sqrt{t}} + \frac{3}{2t+1} \, dt =$

$= \int \frac{3}{\sqrt{t}} e^{\sqrt{t}} dt + \int \frac{3}{2t+1} dt$

$u = \sqrt{t} \quad du = \frac{1}{2\sqrt{t}} dt$ 0.5 $\qquad V = 2t+1$, $dv = 2\,dt$ 0.5

$= \int 6 e^{u} du + \int \frac{1}{2} \frac{3}{v} dv$

$= 6e^{u} + C_1 + \frac{3}{2} \ln|v| + C_2$ 0.5

$= 6e^{\sqrt{t}} + \frac{3}{2} \ln|2t+1| + \boxed{C}$

0.5

(b) $\int_0^7 (y+2)^2 \ln(y+2)\, dy$

we check that the function is well defined in the interval

$y+2 > 0$ if $y \in [0,7]$ ✓ 0.5

$u = y+2$, $du = dy$ 0.5 $\qquad y=0 \quad u=2$, $y=7 \quad u=9$ 0.5

$\int_2^9 u^2 \ln(u)\, du = \left[\frac{1}{3} u^3 \ln(u)\right]_2^9 - \int_2^9 \frac{1}{3} u^2 \, du =$

$$= \frac{1}{3}9^3\ln(9) - \frac{1}{3}2^3\ln(2) - \left[\frac{1}{6}u^3\right]_2^9$$

0.5

$$= \frac{1}{3}9^3\ln(9) - \frac{1}{3}2^3\ln(2) - \frac{1}{6}9^3 - \frac{1}{6}2^3$$

(5)

(a) $|A| = \begin{vmatrix} 0 & 0 & 1 \\ 3 & c & 2 \\ c-2 & 6 & 1 \end{vmatrix} = 1\left(3\cdot 6 - c(c-2)\right)$

$= 1\left(18 - c^2 + 2c\right)$ 2 points

$= \left(-c^2 + 2c - 18\right)$

(b) $|A| = 0$ iff $c^2 - 2c + 18 = 0$

$\frac{\Delta}{4} = \sqrt{1+18} = 19 > 0$ 0.5

$c = 1 \pm \sqrt{19}$

So we have that A is invertible whenever

$c \neq 1 \pm \sqrt{19}$ 0.5

(c)

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ 7 & 1 & 2 & 3 \\ 1 & 6 & 1 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 7 & \boxed{1} & 2 & 3 \\ 2 & 0 & 1 & 1 \\ 1 & 6 & 1 & 3 \end{array}\right) \overset{R_3-6R_1}{\sim} \left(\begin{array}{ccc|c} 7 & 1 & 2 & 3 \\ 2 & 0 & \boxed{1} & 1 \\ -41 & 0 & -11 & -15 \end{array}\right)$$

$R_3 + 11R_2$

$$\left(\begin{array}{ccc|c} 7 & 1 & 2 & 3 \\ 2 & 0 & 1 & 1 \\ -19 & 0 & 0 & -4 \end{array}\right)$$

there are 3 non ~~the~~ zero rows and 3 variable, so there is just one solution

$x = \frac{4}{19}$

$z = 1 - 2 \cdot \frac{4}{19} = \frac{11}{19}$

$y = 3 - \frac{22}{19} - \frac{28}{19} = \frac{57-50}{19} = \frac{7}{19}$

$(x\ y\ z) = \left(\frac{4}{19}, \frac{7}{19}, \frac{11}{19}\right)$

(6) $\frac{\partial}{\partial x} f(xy) = (y^2 - 1) = 0$ 0.5 $y = \pm 1$

$\frac{\partial}{\partial y} f(xy) = 2yx = 0$ 0.5 $x = 0$

There are two critical point $(0,1)$ $(0,-1)$

$\det H = \begin{vmatrix} 0 & 2y \\ 2y & 2x \end{vmatrix} = -4y^2$ 0.5

if $(xy) = (0\,1)$ we have $\det H(0\,1) < 0$
$\det H(0\,-1) < 0$

both points are saddle points

(b) The boundary

$$y^2 = 1 - x^2 \qquad x \in [-1, 1]$$

$$f_{|\partial D}(x,y) = g(x) = x\left(\cancel{1} - x^2 \cancel{-1}\right) = -x^3$$

$$\frac{d}{dx} g(x) = -3x^2 = 0 \qquad x = 0$$

So $g(0) = 0$

$g(-1) = 1$ max

$g(1) = -1$ min

we have that the maximum value taken by f on the boundary is 1 and the minimal is -1

(c) Max on boundary 1

Min on boundary -1

INSIDE

$$f(0,-1) = f(0,1) = 0$$

So the max on min value of f along D ~~is~~ is 1 & the min value is

-1

Solution of Sc with alternative text

$$\begin{cases} +2x \qquad +z = 1 \\ 7y \;+y \;+2z = 8 \\ x \;+6y + z = 3 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ 0 & 8 & 2 & 3 \\ 1 & 6 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} \boxed{1} & 6 & 1 & 3 \\ 2 & 0 & 1 & 1 \\ 0 & 8 & 2 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 6 & 1 & 3 \\ 0 & -12 & \boxed{-1} & -5 \\ 0 & 8 & 2 & 3 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|c} 1 & 6 & 1 & 3 \\ 0 & -12 & -1 & -5 \\ 0 & -16 & 0 & -7 \end{array}\right)$$

we have 3 variables and 3 non-zero rows $\Rightarrow$ 1 solution

$$y = \frac{7}{16}$$

$$z = 5 - 12y = 5 - \cancel{12}^{3}\frac{7}{\cancel{16}_{4}} = \frac{20-21}{4} = -\frac{1}{4}$$

$$x = 3 - 6y - z = 3 - \cancel{6}^{3} \cdot \frac{7}{\cancel{16}_{8}} + \frac{1}{4}$$

$$= \frac{24 - 21 + 2}{8} = \frac{5}{8}$$

The final solution is

$$(x\ y\ z) = \left(\frac{5}{8}, \frac{+7}{16}, -\frac{1}{4}\right)$$