MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Yishao Zhou

Tentamensskrivning i
Linear Algebra and
Learning from Data AN, 7,5 hp
October 26, 2023

You are allowed to bring an A4 sheet (double sides) with whatever you think is important.

---

1. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x) = x^\top Ax$ where $A = \begin{pmatrix} a & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix}$.

   (i) For what values of a is f strictly convex?

   (ii) What is the smallest eigenvalue of H, without further computing, if $a = 2$?

   12 p

2. (i) Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite. Show that $B^\top AB > 0$ if and only if the null space $\mathcal{N}(B) = \{0\}$, where $B \in \mathbb{R}^{n\times k}$.

   (ii) Let $A = xy^\top$ where $x, y \in \mathbb{R}^n$. Show that $\|A\|_2 = \|x\|_2\|y\|_2$. What is $\|A\|_F$?

   (iii) Let A be a square symmetric matrix. Show that $\|A\|_2 = \max\{|\lambda_1|, |\lambda_n|\}$, where λ_1 and λ_n, are the largest and smallest eigenvalues of A, respectively.

   (iv) Let $u, v \in \mathbb{R}^n$ be two unit vector (in $2-$norm) with angle α. Compute $\|uu^\top - vv^\top\|_F^2$. 12 p

3. Assume S is an $n \times n$ symmetric positive definite matrix and C is an $n \times n$ invertible matrix. Show that the $2n \times 2n$ matrix $H = \begin{pmatrix} S & C \\ C^\top & 0 \end{pmatrix}$ is indefinite. How many positive, negative and zero eigenvalues does H have?

   12 p

4. (i) Let $A \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ be such that the Krylov matrix $M = (b, Ab, A^2b, ..., A^{n-1}b)$ is invertible. Show that $M^{-1}AM = C$, where

   $$C = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -p_0 \\ 1 & 0 & 0 & \cdots & 0 & -p_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -p_{n-1} \end{pmatrix}$$

   and the matrix C depends only on A.

   (ii) Find vectors u and v with n-components such that $C = C_{e_2} + uv^\top$, where C_{e_2} is the circulant shift matrix generated by $e_2 = (0, 1, 0, ..., 0)^\top$,

   (iii) Find a QR factorization i.e., $C = Q_1R_1$.

   (iv) Let $C_1 = R_1Q_1$. Repeat the procedure described in the previous two items two more steps. What can be expectedly obtained if we continue this procedure?

   (v) Show that if the polynomial $p(s) = s^n + p_{n-1}s^{n-1} + \cdots + p_0$ has $m(\leq n)$ distinct zeros λ_i, then the matrix C has exactly m linearly independent left eigenvectors $v_i = (1, \lambda_i, \lambda_i^2, ..., \lambda_i^{n-1})$, $i = 1, ..., m$, associated with the eigenvalue λ_i.

   12 p

5. Consider the matrix $T_n(\rho)$ with real components $\rho^{|i-j|}$, $i, j = 1, 2, ..., n$.

   (i) Is it a Toeplitz matrix? Is its inverse a Toeplitz matrix?

   (ii) Discuss for which ρ the matrix is invertible. Find the inverse in these cases.

   (iii) Discuss for which ρ the matrix is positive (semi)-definite.

   (iv) Compute the ℓ^∞-norm of the inverse $T_n^{-1}(\rho)$.

   (v) Give an estimate of the spectral radius of $T_n(\rho)$.

   12 p

You can pick up the graded paper at the student affairs office on Tuesdays at 11:45-12:45.