Solution to Combinatorics' Exam 2024/01/12

Exercise 1

(a) The exponential generating series/function of a_n is

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \sum_0^{\infty} \frac{(n+1)!}{n!} x^n$$

$$= \sum_0^{\infty} (n+1) x^n \quad \text{this is abs conv near 0}$$

$$= \sum_0^{\infty} \frac{d}{dx}(x^{n+1})$$

$$= \frac{d}{dx}\left(\sum_0^{\infty} x^{n+1}\right)$$

$$= \frac{d}{dx}\left(x \cdot \sum_0^{\infty} x^n\right)$$

$$= \frac{d}{dx}\left(x \frac{1}{1-x}\right)$$

$$= \frac{1}{1-x} + x \frac{1}{(1-x)^2}$$

$$= \frac{1-\cancel{x}+\cancel{x}}{(1-x)^2} = \frac{1}{(1-x)^2}$$

(b) $$\sum_0^{\infty} (n+1) 2^{n-1} x^n = \frac{1}{2} \sum_0^{\infty} (n+1)(2x)^n$$

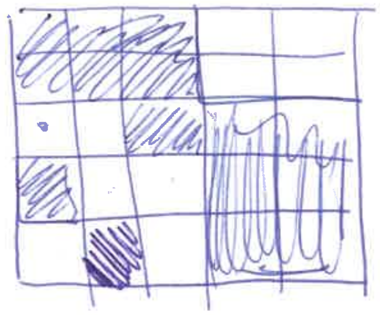
$$= \frac{1}{2}\left(\frac{1}{1-2x}\right)^2$$

no non-exp generating function

(obs $\sum (n+1)! \cdot 2^{n-1} x^n$ does not converge near 0)

Exercise 2

we observe that the rook polynomial of the given chessboard is the same as that of the following



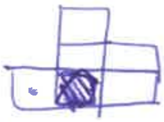
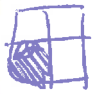
$\leftarrow C_1$

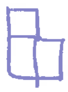
C_2

which is the union of two disjoint chessboards C_1 and C_2

Thus the polynomial is $r(C_1,x)\cdot r(C_2\,x)$

$r(C_1,x) = 1+4x+2x^2$

$r(C_2,x) = r\Big($  $, x\Big) + x\, r\Big($  $, x\Big)$

$= r\Big($  $, x\Big) + x\Big(\cancel{1+2x}$  $, x\Big)$

$+ x\left(1+3x+x^2\right)$

$= r\Big($  $, x\Big) + x\, r\Big($  $\; x\Big) +$

$x\left(1+3x+x^2\right) + x+3x^2+x^3$

$= 1+3x+x^2 + x(1+2x) + x+3x^2+x^3$

$+ x+4x^2+2x^3 =$

$= 1+3x+x^2+x+2x^2+x+3x^2+x^3+x+3x^2+x^3$

$= 1+6x+9x^2+2x^3$

Thus $r(C,x) = (1+4x+2x^2)(1+6x+9x^2+2x^3)$

(b) This is a polynomial of degree 5 so we can place at most 5 rooks

(c) We have to compute the coefficient of degree 3

$$2+9\cdot4+6\cdot2 = 14+36 = 50$$

Exercise 3

(a) $a_1 = 2$ — one blue block / one yellow block

$a_2 =$ 2 blue / 2 yellow / 1 red / 1 blue 1 yellow / 1 yellow 1 blue $= 5$

(b) ~~a_n~~ There are ~~two cases~~ 3 cases

① The last block is red ⇝ a_{n-2} tower

② The last block is blue ⇝ a_{n-1} towers

③ ——— yellow ⇝ a_{n-1} towers

Rule of sum

$$a_n = 2a_{n-1} + a_{n-2}$$

So $a_n - 2a_{n-1} - a_{n-1} = 0$

(b) we find the solution of the homogeneous problem by solving the characteristic equat

$$\lambda^2 - 2\lambda - 1 = 0$$

$$\lambda = 1 \pm \sqrt{1+1} = \begin{matrix} 1+\sqrt{2} \\ 1-\sqrt{2} \end{matrix}$$

So we have that

$$a_n = A(1+\sqrt{2})^n + B(1-\sqrt{2})^n$$

$$a_0 = A + B = 1$$

$$a_1 = A(1+\sqrt{2}) + B(1-\sqrt{2}) = 2$$

$$\begin{cases} A + B = 1 \\ 0 + B[\cancel{1}-\sqrt{2}-\cancel{1}-\sqrt{2}] = 2-(1+\sqrt{2}) \end{cases}$$

$$\overset{\shortparallel}{1-\sqrt{2}}$$

$$A + B = 1$$

$$-2\sqrt{2}B = 1-\sqrt{2}$$

$$B = \frac{\sqrt{2}-1}{2\sqrt{2}}$$

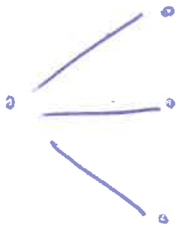
$$A = 1 - \frac{\sqrt{2}-1}{2\sqrt{2}} = \frac{2\sqrt{2}-\sqrt{2}+1}{2\sqrt{2}} = \frac{1+\sqrt{2}}{2\sqrt{2}}$$

Thus we have that

$$a_n = \left(\frac{1+\sqrt{2}}{2\sqrt{2}}\right)(1+\sqrt{2})^n + \left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)(1-\sqrt{2})^n$$

Exercise 4

(a) $K_{1,3}$



$LK_{1,3}$

$= K_3$

(b) $\deg(e)$ in LG is the number of edges adjacent to e thus it is the number of edges who have a vertex in common with e

there are $\deg(v)-1$ other edges ~~spanning~~ containing v

and $\deg(w)-1$ other edges containing w

$$\deg(e) = \deg(v) + \deg(w) - 2$$

(c) if v is a vertex of $\deg(v) = d$ then it is ~~adjacent~~ contained in d edges. and every ~~po~~ unordered pair of them yield an edge in LG

$\Rightarrow$ so it contribute to $\binom{d}{2}$ edges of the line graph.

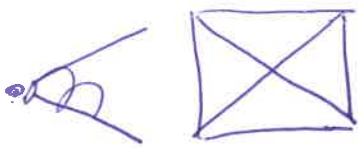
$$|E(LG)| = \sum_{v\in V} \binom{\deg(v)}{2}$$

$$= \sum_{v\in V} \frac{\deg(v)(\deg(v)-1)}{2}$$

$$= \sum_{v\in V} \frac{\deg(v)^2}{2} - \frac{1}{2}\sum_{v\in V} \frac{\deg v}{2}$$

$$= \sum_{v\in V} \frac{\deg(v)^2}{2} - \frac{2}{2}|E(G)|$$

(d) If G has an Euler circuit then all its vertices have even degree but then, by (a) all its edges have even degree ~~and thus~~ in LG and thus LG has an euler circuit



K_4 does not have an Euler circuit because it has vertices of odd degree. On the other side this is 3 regular so the degree of its edges in LK_4 is $3+3-2=4$ which means that LK_4 has an Euler circuit.

Alternative the example in (a)

Exercise 5

(a) $\{s v\}$ $\{s t\}$ $\{v w\}$ $\{u w\}$

$w(T)=10$

(b) $\{u w\}$ $\{s v\}$ $\{s t\}$ $\{u v\}$

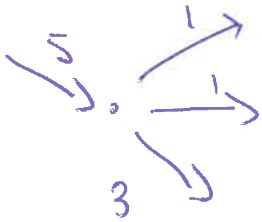
$w(T)=10$

Exercise 6

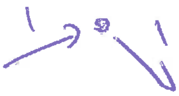
(a) we have to check the balancing condition

✓

✓



$5=1+1+3$



$1=1$

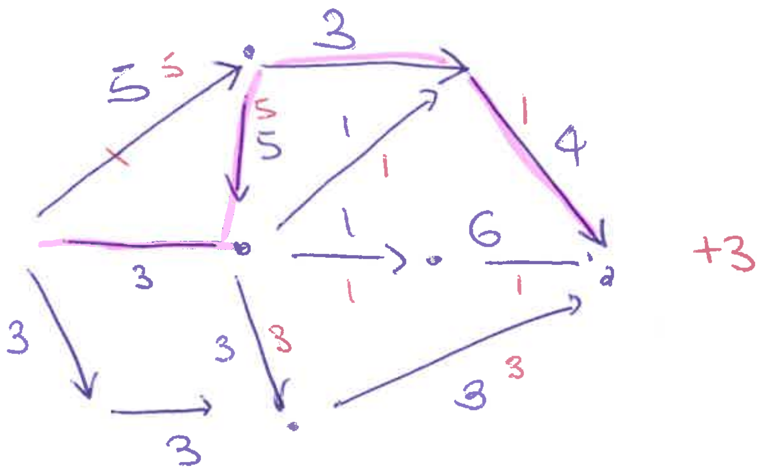
✓

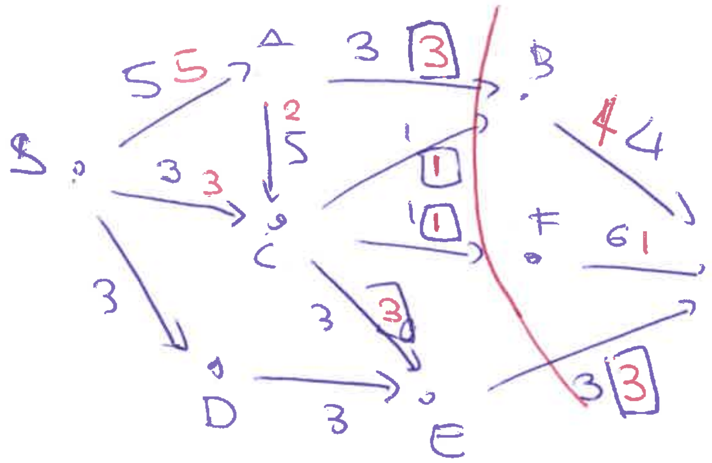
$1=1$

✓

$3=3$

The value of the flow is 5





no other augmentin path exist

~~we pro~~ the value of the flow is $\boxed{8}$

(c) $P = \{S, D, E, C, A\}$

$P^c = \{T, B, F\}$

$c(P, P^c) = 3 + 1 + 1 + 3 = 8$

Since $c(P, P^c) = Val(f)$ we have that the flow is maximal & the cut minimal