MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Yishao Zhou

Tentamensskrivning i
Optimering AN, 7,5 hp
January 4, 2024

You are allowed to bring an A4 sheet (double sides) with whatever you think is important.
Write down on your cover page how many points p_h you collected from the homework assignments.
You must motivate well your arguments.

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1. (i) Give two definitions of a convex function and show that they are equivalent.
   (ii) Is the function $f(x) = \sqrt[3]{x_1 x_2 x_3}$ convex or concave on $\mathbb{R}^3_{++}$?
   (iii) Is the set $\{x \in \mathbb{R}^3_{++} : -\sqrt[3]{x_1 x_2 x_3} < 1\}$ a convex set?
   (iv) Assume that f is a convex function. Show that d is a descent direction of f at x if and only if $f'(x; d) < 0$. Does the result hold without convexity of f? 12 p

2. Consider the following problem:

$$\begin{array}{rl} \max & 3x_1 + 4x_2 + 5x_3 + 6x_4 \\ \text{s.t.} & 4x_1 + 5x_2 + 6x_3 + 7x_4 \leq 17 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\}. \end{array}$$

   (i) Determine its dual objective function (explicitly) and the corresponding dual problem.
   (ii) Determine the duality gap.
   (iii) Solve the problem based on above analysis. 12 p

3. Consider the optimization problem: min $f(x)$ subject to $g_i(x) \leq 0$ for $i = 1, ...m$.

   (i) Show that verifying whether a point $\bar{x}$ is a KKT point is equivalent to finding a vector u such that $A^\top u = c, u \geq 0$ where $A \in \mathbb{R}^{m\times n}$.
   (ii) Use this together with the Simplex Phase I to check whether $\bar{x} = (2, 1)^\top$ is a KKT point to the following problem:

$$\begin{array}{rl} \text{Minimize} & (x_1 - 3)^2 + (x_2 - 2)^2 \\ \text{subject to} & x_1^2 + x_2^2 \leq 5 \\ & x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0. \end{array}$$

   12 p

4. Let $S = \{x \in \mathbb{R}^2_+ : x_1^2 + x_2^2 \leq 5, x_1 + 2x_2 \leq 4\}$ and $y = (3, 2)^\top$. Find the minimum distance from y to S, the unique minimizing point and a separating hyperplane. 12 p

5. Let $\mathcal{C}_n := \{M \in \mathcal{S}^n : x^\top M x \geq 0, \forall x \in \mathbb{R}^n_+\}$ and let A_{JK} be submatrix of A with rows indexed by the (nonempty) index set J and columns by the (nonempty) index set K, and $\mathbf{e}$ be an all one column vector, and $|J|$ be the cardinality of the set J.

   (i) Show that the standard quadratic optimization (SQO) problem,

$$\min_{x\in\Delta_n} \; x^\top Q x$$

   with $Q \in \mathcal{S}^n$, and $\Delta_n := \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$ can be rewritten as

$$\max_{t\in\mathbb{R}} \{t : Q - t\mathbf{e}\mathbf{e}^\top \in \mathcal{C}_n\}.$$

(ii) Show that exactly one of the following two systems has a solution.
a. $Ax \geq 0, x \geq 0$, and $c^\top x > 0$.
b. $A^\top y \geq c$ and $y \leq 0$.
(If you are not able to prove it take it as granted.)

(iii) Show that $M \in \mathcal{C}_n$ if and only if for all $J \subseteq \{1, ..., n\}$, $J \neq \emptyset$, the following system has a solution: $M_{JJ}x_J \geq 0$, $x_J \geq 0$, $\mathbf{e}_{|J|}^\top x_J = 1$. (If you are not able to prove it take it as granted.)

(iv) Show that the SQO can be reformulated as the following LP problem

$$\max \left\{ t : Q_{JJ}x_J - t\mathbf{e}_{|J|}^\top \geq 0, x_J \geq 0, \mathbf{e}_{|J|}^\top x_J = 1, \forall J \subseteq \{1, ..., n\} \right\}.$$

12 p

You have finished the exam if your homework $p_h \geq 30$ where p_h is the points you gained from homework assignments. Continue otherwise.
Solve 1 or 2 or 3 of the following problems according to the ordering $29 \geq p_h \geq 20$, or $19 \geq p_h \geq 10$, or $p_h \leq 9$.

6. Consider the distance function to a closed convex set C:

$$d(x, C) = \min_{y \in C} \|y - x\|_2.$$

(i) Show that it is a convex function.

(ii) Find its subdifferential $\partial d(x, C)$.

13 p

7. Consider the support function S on a nonempty bounded convex set C:

$$S(y) = \sup_{x \in C} y^\top x.$$

(i) Show that it is a convex function.

(ii) Let $C := \{x : (x - \bar{x})^\top Q(x - \bar{x}) \leq b\}$ where $b > 0$, $Q \in \mathcal{S}^n_{++}$ and $\bar{x} \in \mathbb{R}^n$. Determine $S_C(y)$.

(iii) Let C be a cone in $\mathbb{R}^n$. Determine $S_C(y)$.

13 p

8. Consider the primal problem

$$\begin{array}{lll} (P) & \min & f(x) \\ & \text{s.t.} & g_i(x) \leq 0, \ i = 1, ..., m, \\ & & h_j(x) = 0, \ j = 1, ..., p, \\ & & x \in X \subseteq \mathbb{R}^n. \end{array}$$

Assume that strong duality holds, x^* is primal optimal, (λ^*, μ^*) is dual optimal.

(i) Show that $\lambda_i g_i(x^*) = 0$ for $i = 1, ..., m$.

(ii) Solve first the following LP problem graphically then its dual optimal solution using the preceding statement.

$$\begin{array}{ll} \min & -2x_1 - x_2 \\ \text{s.t.} & x_1 + x_2 \leq 3, \\ & x_1 - x_2 \geq -1 \\ & x_1, x_2 \geq 0. \end{array}$$

13 p

You can pick up the graded paper at the student affairs office on Tuesdays at 11:45-12:45.