MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Jonas Bergström

Tentamensskrivning i
Topologi MM7052
7,5 hp
2024-01-24

(1) (a) [1 pt] Show that the collection $\mathcal{B} = \{[a, b) \subset \mathbb{R} : a < b\}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}$.
(b) [1 pt] What are the limit points of the set $(0, 1)$ with respect to $\mathcal{T}$?
(c) [1 pt] What are the connected components of $\mathbb{R}$ with respect to $\mathcal{T}$?
(d) [1 pt] Show that $\mathbb{R}$ with respect to $\mathcal{T}$ is not compact.
(e) [1 pt] Show that $\mathbb{R}$ with respect to $\mathcal{T}$ is not second countable.

(2) Let X be any non-empty topological space, and define $CX = (X \times I)/(X \times \{0\})$ which is called the cone on X.
(a) [3 pts] Show that CX is contractible.
(b) [2 pts] For any $n \geq 0$, show that $C\mathbb{S}^n$ is homeomorphic to $\overline{\mathbb{B}}^{n+1}$.

(3) We define a group action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ by putting $(m, n) \cdot (x, y) = (x+m, y+n)$ for any $(m, n) \in \mathbb{Z}^2$ and $(x, y) \in \mathbb{R}^2$.
(a) [2 pt] Show that this is a covering space action (also called a properly discontinuous action).
(b) [3 pt] Show that $\mathbb{R}^2/\mathbb{Z}^2$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$.

(4) Let $q : E \to X$ be a covering map.
(a) [3 pt] Show that if X is Hausdorff then E is Hausdorff.
(b) [2 pt] Show that if q is proper then $q^{-1}(x)$ consists of finitely many points for all $x \in X$.

(5) [5 pts] Let $\overline{\mathbb{B}}^2$ denote the closed disc with boundary $\mathbb{S}^1$. Fix a point $p \in \mathbb{S}^1$. Compute the fundamental group of the union $(\mathbb{S}^1 \times \mathbb{S}^1) \cup (\overline{\mathbb{B}}^2 \times \{p\})$ inside $\overline{\mathbb{B}}^2 \times \mathbb{S}^1$.

(6) [5 pts] Prove the following theorem:

Theorem 1. *(Homotopy invariance of π_1.) If $\varphi : X \to Y$ is a homotopy equivalence, then for any point $p \in X$, $\varphi_* : \pi_1(X, p) \to \pi_1(Y, \varphi(p))$ is an isomorphism.*