MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Sofia Tirabassi

Exam in
Combinatorics
7.5 hp
February 15th, 2024

Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- A maximum score of 30 points can be achieved. A score of at least 15 points will ensure a pass grade.

GOOD LUCK!

1. **Rook polynomials:** Suppose we have people A, B C and D, jobs a, b c and d, and the following qualifications:

    - a can be performed by A or D
    - b can be performed by B or D
    - c can be performed by B or C
    - d can be performed by A or B.

    (a) (2 points) Describe the allocating problem by using a 4×4 chessboard with forbidden positions.

    (b) (2 points) Compute the rook polynomial of this chessboard.

    (c) (1 point) Say whether it is possible to assign the four jobs to four different persons. If yes, in how many ways can this be done?

2. **Recursion:** (4 points) Use the **method of generating functions'** to solve the following recursion:

$$a_{n+2} - 3a_{n+1} + 2a_n = 3^{n+2}$$

with initial conditions $a_0 = 1$ and $a_1 = 6$. You may use the following identity

$$\frac{1}{(1-x)(1-2x)(1-3x)} = \frac{1}{2}\frac{1}{1-x} - 4\frac{1}{1-2x} + \frac{9}{2}\frac{1}{1-3x}$$

3. **Graphs** Consider the graph G pictured in Figure 1.

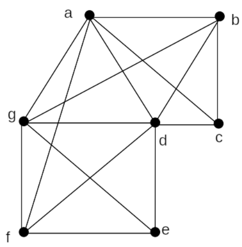

Figure 1: Graph

    (a) (2 points) Determine whether G is planar.

    (b) (2 points) Draw the graph H, obtained from G by collapsing first $\{a, d\}$ and then $\{d, c\}$.

    (c) (2 points) Compute the chromatic polynomial of H.

4. **Shortest Path:** Consider the weighted directed graph in Figure 2.

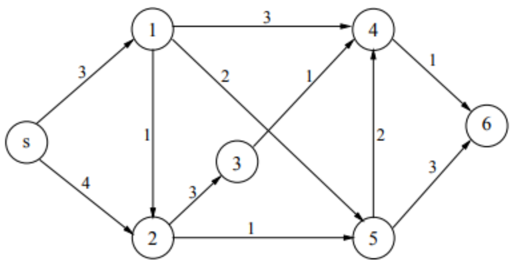

Figure 2: Weighted Graph

(a) (3 points) Find the shortest path from s to all other vertices using Dijstra's algorithm. Show clearly all the iteration, possibly with the use of tables.

(b) (1 point) Suppose now that the weight of $(3,4)$ is changed to -2, show that the algorithm does not yield anymore the shortest paths.

(c) (1 point) Explain why the easy fix of adding to all the weight in (b) 2 in such a way that all the weight are positive, might not give a shortest path in the original graph.

5. **Latin squares** Let p be a prime and $a \in \mathbb{Z}/p\mathbb{Z}^*$.

(a) (1 point) Consider the $n \times n$ table such that

$$L(i,j) = ia + j \mod p.$$

Show that this yield a standard Latin square of order n (using the order $0 < 1 < \cdots < p-1$).

(b) (1 point) A Latin square is said to be diagonal if it has no repeated entry in the two diagonals. Show that there are no standard diagonal Latin squares of order 3.

(c) (2 points) Show that if $a \neq 1, p-1$, then the Latin square constructed in (a) is diagonal.

(d) (1 point) Construct a diagonal Latin square of order 5.

6. **Transport Networks:** Consider the transport network in Figure 3, where s is the source and t is the sink.

(a) (2 points) Find a flow with the maximum value for the network.

(b) (3 points) Show that the flow in (a) is indeed maximal by giving a cut with the same capacity.

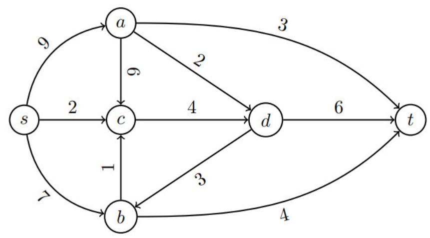

Figure 3: Network