MATEMATISKA INSTITUTIONEN<br>STOCKHOLMS UNIVERSITET<br>Avd. Matematik<br>Examinator: Sofia Tirabassi

Make up assignment

## MM5020 Abstract Algebra

7.5 hp

March 8th, 2024

## Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers. A correct answer without proper justification will not award full points.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- A maximum score of 30 points can be achieved.

1. Let $G$ be a group and $N$ and $H$ two of its subgroups. For each of the following statements, determine if it is true or false. Give a brief justification or a counterexample.
(a) (2 pts) Let $N<H<G$ such that $N \triangleleft H$ and $H \triangleleft G$, then $N \triangleleft G$.
(b) (2 pts) Let $N \triangleleft G$ with $N$ abelian and $G / N$ cyclic, then $G$ is abelian.
2. Recall that the direct product of two groups $G_{1} \times G_{2}$ is the given by the set $G_{1} \times G_{2}$ with operation $(x, y)(z, t)=(x z, y t)$. Let $G$ be a group with $N$ and $M$ two subgroups.
(a) (2 pts) Show that, if both $N$ and $M$ are normal in $G$, then $N M$ is a normal subgroup of $G$.
(b) (2 pts) Show that if $N$ and $M$ are normal in $G, N M=G$ and $N \cap M=\{e\}$, then $G$ is isomorphic to $N \times M$. (Hint: First shoe that $m n=n m$ for all $m \in M$ and all $n \in N$.
(c) ( 2 pts ) Show that an abelian group of order $75=3 \cdot 5^{2}$ is either cyclic or isomorphic to $\mathbb{Z} / 3 \mathbb{Z} \times$ $(\mathbb{Z} / 5 \mathbb{Z} \times \mathbb{Z} / 5 \mathbb{Z})$
3. Let $G$ act on a set $X$ and set

$$
X^{G}:=\{x \in X \mid g \cdot x=x \text { for all } g \in G\}
$$

(a) (2 pts) Show that $x \in X^{G}$ if, and only if the orbit of $x$ consists of one element.
(b) (3 pts) Suppose that $G$ is a $p$-group, show that

$$
|X| \equiv\left|X^{G}\right| \quad \bmod p
$$

4. Show the following statements.
(a) (2 pts) If $G$ has order $165=3 \cdot 5 \cdot 11$ and $\operatorname{Syl}_{5}(G)=\{P\}$ then $P \leq Z(G)$.
(b) (2 pts) There is no simple group of order $351=3^{3} \cdot 13$.
5. Let $R$ be a unitary commutative ring. We say that $x \in R$ is nilpotent if, and only if, $x^{n}=0$ for some $n \in \mathbb{Z}, n \geq 0$.
(a) (3 pts) Let

$$
\mathfrak{R}:=\{x \in R \mid x \text { is nilpotent }\} .
$$

Show that this is an ideal of $R$ (Hint: The binomial theorem works in every commutative ring).
(b) ( 2 pts ) Show that $\mathfrak{R}$ is contained in the intersection of all prime ideals of $R$.
6. Let $\mathbb{Q}(2 \sqrt{3}-\sqrt{5})$ (respectively $\mathbb{Q}(\sqrt{3}, \sqrt{5}))$ be the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and $2 \sqrt{3}-\sqrt{5}$ (respectively $\sqrt{3}$ and $\sqrt{5}$ ).
(a) (2 pts) Show that $x^{4}-34 x^{2}+49$ is the minimal polynomial of $2 \sqrt{3}-\sqrt{5}$ over $\mathbb{Q}$.
(b) (2 pts) Compute $[\mathbb{Q}(2 \sqrt{3}-\sqrt{5}): \mathbb{Q}]$ and find a basis $\mathbb{Q}(2 \sqrt{3}-\sqrt{5})$ over $\mathbb{Q}$.
(c) $(2 \mathrm{pts})$ Show that $\mathbb{Q}(\sqrt{3}, \sqrt{5})=\mathbb{Q}(2 \sqrt{3}-\sqrt{5})$.

