MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
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Make up assignment
MM5020 Abstract Algebra
7.5 hp
March 8th, 2024

Please read carefully the general instructions:

- During the exam any textbook, class notes, or any other supporting material is forbidden.
- In particular, calculators are not allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. Justify your answers. A correct answer without proper justification will not award full points.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- A maximum score of 30 points can be achieved.

GOOD LUCK!

1. Let G be a group and N and H two of its subgroups. For each of the following statements, determine if it is true or false. Give a brief justification or a counterexample.

   (a) (2 pts) Let $N < H < G$ such that $N \lhd H$ and $H \lhd G$, then $N \lhd G$.

   (b) (2 pts) Let $N \lhd G$ with N abelian and G/N cyclic, then G is abelian.

2. Recall that the direct product of two groups $G_1 \times G_2$ is the given by the set $G_1 \times G_2$ with operation $(x, y)(z, t) = (xz, yt)$. Let G be a group with N and M two subgroups.

   (a) (2 pts) Show that, if both N and M are normal in G, then NM is a normal subgroup of G.

   (b) (2 pts) Show that if N and M are normal in G, $NM = G$ and $N \cap M = \{e\}$, then G is isomorphic to $N \times M$. (***Hint:*** *First shoe that $mn = nm$ for all $m \in M$ and all $n \in N$.*

   (c) (2 pts) Show that an abelian group of order $75 = 3 \cdot 5^2$ is either cyclic or isomorphic to $\mathbb{Z}/3\mathbb{Z} \times (\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z})$

3. Let G act on a set X and set

$$X^G := \{x \in X \mid g \cdot x = x \text{ for all } g \in G\}.$$

   (a) (2 pts) Show that $x \in X^G$ if, and only if the orbit of x consists of one element.

   (b) (3 pts) Suppose that G is a p-group, show that

$$|X| \equiv |X^G| \mod p.$$

4. Show the following statements.

   (a) (2 pts) If G has order $165 = 3 \cdot 5 \cdot 11$ and $\mathrm{Syl}_5(G) = \{P\}$ then $P \leq Z(G)$.

   (b) (2 pts) There is no simple group of order $351 = 3^3 \cdot 13$.

5. Let R be a unitary commutative ring. We say that $x \in R$ is nilpotent if, and only if, $x^n = 0$ for some $n \in \mathbb{Z}$, $n \geq 0$.

   (a) (3 pts) Let

$$\mathfrak{N} := \{x \in R \mid x \text{ is nilpotent}\}.$$

   Show that this is an ideal of R (***Hint:*** *The binomial theorem works in every commutative ring*).

   (b) (2 pts) Show that $\mathfrak{N}$ is contained in the intersection of all prime ideals of R.

6. Let $\mathbb{Q}(2\sqrt{3} - \sqrt{5})$ (respectively $\mathbb{Q}(\sqrt{3}, \sqrt{5})$) be the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and $2\sqrt{3} - \sqrt{5}$ (respectively $\sqrt{3}$ and $\sqrt{5}$).

   (a) (2 pts) Show that $x^4 - 34x^2 + 49$ is the minimal polynomial of $2\sqrt{3} - \sqrt{5}$ over $\mathbb{Q}$.

   (b) (2 pts) Compute $[\mathbb{Q}(2\sqrt{3} - \sqrt{5}) : \mathbb{Q}]$ and find a basis $\mathbb{Q}(2\sqrt{3} - \sqrt{5})$ over $\mathbb{Q}$.

   (c) (2 pts) Show that $\mathbb{Q}(\sqrt{3}, \sqrt{5}) = \mathbb{Q}(2\sqrt{3} - \sqrt{5})$.