MM5020 - Solution to exam 20240417

Exercise 1

(a) This is false consider $H = \langle (123) \rangle \trianglelefteq S_3$

$$S_3 / H \simeq \mathbb{Z}/2\mathbb{Z} \quad \text{abelian}$$

but $Z(S_3) = \{id\} \not\supseteq H$

(b) This is false let $\frac{a}{b} \in \mathbb{Q}$. We can suppose that $b > 0$

$$b\left(\frac{a}{b} + \mathbb{Z}\right) = a + \mathbb{Z} = \mathbb{Z}$$

$\Rightarrow$ The order of $\left(\frac{a}{b} + \mathbb{Z}\right) \leq b$ and so it cannot be infinite

Exercise 2

(a) Let $g \in G$ and consider $\sigma_g : G \longrightarrow G$, $h \longmapsto ghg^{-1}$ $\in \mathrm{Aut}(G)$

If H is characteristic then $\sigma_g(H) = H$ which means $gHg^{-1} = H$ thus H is normal

(b) Let $g \in G$ since $N \trianglelefteq G$ we have that conjugation by g induces an isomorphism $\sigma_{g|N} : N \longrightarrow N$, $n \longmapsto gng^{-1}$

If H char N then $\sigma_{g|N}(H) = H$ that is for all $h \in H$

$$ghg^{-1} = \sigma_{g|N}(h) \in H \quad \rightarrow \quad gHg^{-1} = H \quad \text{and} \quad H \trianglelefteq G$$

(c) Let $P \in \mathrm{Syl}_p(G)$. By definition

$$N_G(P) = \{ g \in G \mid gPg^{-1} = P \}$$

In particular $P \trianglelefteq N_G(P) < G$

We deduce that $P \in \mathrm{Syl}_p(N_G(P))$. In particular P is the unique subgroup of $N_G(P)$ of its order which easily implies that P char $N_G(P)$

Exercise 3 :

(a) Since the action is transitive it means there is just one orbit of size n

Observe that G_1 is the stabilizer of 1. By the orbit stabilizer theorem

$$n = \frac{|G|}{|G_1|} = [G : G_1]$$

(b) We want to use (a) and show that $G_1 = \{id\}$

let $g \in G_1$ and $k \in \{1 \dots n\}$. Since the action is transitive

we can write $k = h \cdot 1$ for some $h \in G$

Now $g \cdot k = gh \cdot 1 = hg \cdot 1 = h \cdot 1 = k$ $\Rightarrow$ g acts as the identity

$\Rightarrow G_1 = \{g\}$

Exercise 4

(a) $2^3 \cdot 3 \cdot 13$

$n_{13} \equiv 1 \mod 13$ and divides 24

$\Rightarrow n_{13} = 1$ and G has a normal 13-Sylow group

(b) $2^3 \cdot 5^2$

$n_5 \equiv 1 \mod 5$ $n_5 | 8$ $\Rightarrow$ $n_5 = 1$ G has a normal 5 sylow group

Exercise 5

(a) given $x, y \in J$ we have to show that

$x + y \in J$

$ax \in J$ for all $a \in R$

Let m a maximal ideal $m \ni x, y \rightarrow m \ni x+y$ so $x+y \in \cap m$

$m \ni x \rightarrow m \ni ax$ for all $a \in R \Rightarrow ax \in \cap m$

(b) Let $a \in R$ be such that $b = (1-ax)$ is not a unit. Consider (b) the ideal generated by b then $\exists$ m maximal $m \supseteq (b)$ If $x \in m$

then $ax \in m$ and $1 = 1 - ax + ax \in m$ which is a contradiction

(c) Suppose that $x \notin J$, then there is a maximal ideal such that

$x \notin m$. The quotient R/m is a field and $x+m \neq 0$

thus $\exists$ $a+m$ such that $1+m = (a+m)(x+m) = ax+m$

$\Rightarrow$ $1-ax \in m$ that is $1-ax$ is not a unit

Exercise 6

(a) $p(x) = x^3+x+1$ $\quad p(0)=1 \quad p(1)=3 \quad p(2)=1 \quad p(3)=1 \quad p(4)=4$

So p has no roots since $\deg p \leq 3 \Rightarrow p$ is irreducible

$\Rightarrow$ (p) is prime since $k[x]$ is UFD

$\Rightarrow$ (p) is maximal since $k[x]$ is PID

$\Rightarrow$ $\mathbb{Z}/5\mathbb{Z}[x]/(p)$ is a field

(b) We already know from (a) that p is irreducible thus we have to show that $p(\alpha)=0$

$$p(\alpha) = \alpha^3+\alpha+1 = (x+(p))^3 + (x+(p)) + (1+(p))$$
$$= x^3+(p) + x+(p) + 1+(p)$$
$$= x^3+x+1+(p) = p(x)+(p) = 0$$

we have that $1, \alpha, \alpha^2$ gives a basis of $\mathbb{Z}/5\mathbb{Z}(\alpha)/\mathbb{Z}/5\mathbb{Z}$

(c) $$0 = \alpha(\alpha^3+\alpha+1) = \alpha^4+\alpha^2+\alpha \Rightarrow \alpha^4+\alpha = -\alpha^2 = 4\alpha^2$$

$$0 = \alpha^2(\alpha^3+\alpha+1) = \alpha^5+\alpha^3+\alpha^2 = \alpha^5-\alpha-1+\alpha^2$$
$$\alpha^5 = -\alpha^2+\alpha+1 = 4\alpha^2+\alpha+1$$