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Suggested solutions

Exam: Introduction to Finance Mathematics (MT5009), 2024-05-23

Problem 1

(A) Basic calculations give

$$\begin{aligned} V(0) &= e^{-r}C + e^{-2r}(F + C) \\ &= 99.11. \end{aligned}$$

(B) The value of the bond $t \leq 10$ years from now is

$$B(t, 10) = e^{-r(10-t)}.$$

The value of the bond in two years is therefore $B(2, 10) = e^{-0.1(10-2)} = 0.4493$.

We are also asked to find t so that $B(t, 10) = 0.9$. This corresponds to solving

$$t = 10 + \frac{\ln(0.9)}{r} = 8.9464$$

i.e., the bond will be worth 0.9 in 8.9464 years.

Problem 2

(A) A portfolio strategy corresponds to a plan for how a financial portfolio should be rebalanced (updated) over time given the development of the prices of a set of assets. A portfolio strategy is said to replicate a derivative if at each time and for each possible value of the asset(s) underlying the derivative it holds that the value of the derivative is equal to that of the replicating portfolio strategy.

(B) If a portfolio with value $V(0)$ satisfies the following three conditions then it is an arbitrage opportunity (Capinski & Zastawniak ch. 1.2):

(i) $V(0) = 0$,
(ii) $\mathbb{P}(V(1) \geq 0) = 1$, and
(iii) $\mathbb{P}(V(1) > 0) > 0$.

Problem 3

(A) Note that

$$\begin{aligned} p_* &= \frac{R-D}{U-D} = 0.75 \\ S^{uu} &= S(0)(1+U)^2 = 121, \\ S^{ud} &= S(0)(1+U)(1+D) = 99, \\ S^{dd} &= S(0)(1+D)^2 = 81. \end{aligned}$$

The risk-neutral valuation formula for the call is

$$\begin{aligned} &C_E(0) \\ &= \frac{1}{(1+R)^2} E_* \left[(S(2) - X)_+ \right] \\ &= \frac{1}{(1+R)^2} \left[p_*^2 (S^{uu} - X)_+ + 2p_*(1-p_*)(S^{ud} - X)_+ + (1-p_*)^2 (S^{dd} - X)_+ \right]. \end{aligned} \tag{1}$$

By plugging in the numbers and using basic calculations we find

$$C_E(0) = 10.7143.$$

(B) Using our usual notation (see Capinski & Zastawniak) we find

$$\begin{aligned} C_E^u &= \frac{1}{(1+R)} \left[p_*(S^{uu} - X)_+ + (1-p_*)(S^{ud} - X)_+ \right] = 15 \\ C_E^d &= \frac{1}{(1+R)} \left[p_*(S^{ud} - X)_+ + (1-p_*)(S^{dd} - X)_+ \right] = 0 \\ C_E^{uu} &= (S^{uu} - X)_+ = 21 \\ C_E^{ud} &= (S^{ud} - X)_+ = 0 \\ C_E^{dd} &= (S^{dd} - X)_+ = 0. \end{aligned}$$

Using this we can describe the replicating strategy for the option as follows:

The replicating portfolio formed at time 0 is given by

$$\begin{aligned} x(1) &= \frac{C_E^u - C_E^d}{S^u - S^d} = 0.75 \\ y(1) &= \frac{C_E^u - x(1)S^u}{1+R} = -64.2857. \end{aligned}$$

The replicating portfolio formed at time 1 in case $S(1) = S^u$ is given by

$$\begin{aligned} x^u(2) &= \frac{C_E^{uu} - C_E^{ud}}{S^{uu} - S^{ud}} = 0.9545 \\ y^u(2) &= \frac{C_E^{uu} - x^u(2)S^{uu}}{1+R} = -90.0000. \end{aligned}$$

The replicating portfolio formed at time 1 in case $S(1) = S^d$ is given by

$$\begin{aligned} x^d(2) &= \frac{C_E^{ud} - C_E^{dd}}{S^{ud} - S^{dd}} = 0, \\ y^d(2) &= \frac{C_E^{ud} - x^d(2)S^{ud}}{1+R} = 0. \end{aligned}$$

Problem 4

(A) The weights in the minimum variance portfolio is given by s_0 (for share 1) and $1 - s_0$ (for share 2) where s_0 is given by the formula in the question. Plugging in numbers gives the portfolio weights

$$\begin{aligned} s_0 &= 0.5397 \\ 1 - s_0 &= 0.4603. \end{aligned}$$

(B) This is shown in Capinski & Zastawniak p. 64-65. Note that a complete solution includes (i) observing that $\rho_{12} < 1$ implies that $\sigma_1^2 + \sigma_2^2 - 2c_{12}$ is non-zero which rules out that the derivation of s_0 includes dividing by zero, and (ii) some argument as to why the critical point found (by setting the derivative of the variance with respect to s equal to zero) corresponds to a (global) minimum.

Problem 5

The derivative gives you the fixed amount 10 in case the price of the underlying is smaller than 1 at maturity, and 0 otherwise (a kind of *cash-or-nothing option*).

With $T = 2$, and the usual notation, we find (see around page 214 in Capinski & Zastawniak)

$$\begin{aligned} V(0) &= e^{-rT} E_* \left(10 I_{\{S(T)<1\}}\right) \\ &= 10e^{-2r} E_* \left(I_{\{S(0)e^{(r-\frac{1}{2}\sigma^2)2+\sigma W_*(2)}<1\}}\right) \\ &= 10e^{-2r} E_* \left(I_{\{W_*(2)<(ln(1/S(0))-2(r-\frac{1}{2}\sigma^2))/\sigma\}}\right). \end{aligned}$$

Let $Z \sim N(0,1)$ so that $W_*(2)$ and $\sqrt{2}Z$ have the same distribution. With some rewriting and basic probability we then find

$$\begin{aligned} V(0) &= 10e^{-2r} E_* \left(I_{\{Z<(ln(1/S(0))-2(r-\frac{1}{2}\sigma^2))/(\sqrt{2}\sigma)\}}\right) \\ &= 10e^{-2r} P_* \left(Z < (ln(1/S(0)) - 2(r - \frac{1}{2}\sigma^2))/(\sqrt{2}\sigma)\right). \end{aligned}$$

Denoting the distribution function of a standard normal random variable by N we find the pricing formula

$$V(0) = 10e^{-2r} N \left(\frac{ln(1/S(0)) - 2(r - \frac{1}{2}\sigma^2)}{\sqrt{2}\sigma}\right).$$