Introduction to Real Analysis

Lecture 1

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Logistic Information

- **Book:** Walter Rudin, *Principle of Mathematical Analysis (3:rd ed)*

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- **Time and Place:** Variable

Place : Zoom.

Time : meeting at the beginning of a week every week.

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- **Book:** Walter Rudin, *Principle of Mathematical Analysis (3:rd ed)*
- **Time and Place:**
- **Final Exam:** Look at the course web page / schema.su.se

Grading

- The grade of the course (G) will be over 30 points ($0 \leq G \leq 30$).
- All students should take a written exam, which will consist of up to 5 problems. The score obtained in the written exam (W) will give a maximum of 24 points ($0 \leq W \leq 24$).
- Those students that obtain a score greater than 21 in the written exam ($21 \leq W \leq 24$) have the right to take an oral exam. The score of the oral exam (O) will give a maximum of 6 points ($0 \leq O \leq 6$).
- During the course three homework will be released, so that each student can obtain bonus points (B). Each homework will be graded over 1 point, so that a maximum of 3 bonus points can be obtained ($0 \leq B \leq 3$).
- If a student obtains a score smaller than 21 in the written exam ($0 \leq W < 21$), the grade will be given by $G = W + B$. If a student obtains a score greater than 21 in the written exam ($21 \leq W \leq 24$), the grade will be given by $G = \min(W + B, 24) + O$.

Section 1
Lecture 1

Lecture Plan

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- Ordered sets (Rudin 1.5-1.11)
- Ordered fields (Rudin 1.12-1.18)
- The Real numbers (Rudin 1.13-1.22)
- Extended Real numbers (Rudin 1.23)
- The Complex numbers (Rudin 1.24-1.35)
- Cardinality (Rudin 2.4-2.14) ← Combinatorics

Section 2
Ordered Sets

Orders

(Totally ordered sets)

Given a set S an order on S is a relation $<$ on S such that

O1 Given x and y in S, one, and only one of the following is true

$$x = y, \quad x < y, \quad y < x$$

O2 $<$ is transitive.

The pair $(S, <)$ is called ordered set.

Non example $S = \{1, 2, 3\}$

$(\mathcal{P}(S), \subseteq)$ is not ordered

$\{1,2\}$ $\{2,3\}$ not =

$\{1,2\} \not\subseteq \{2,3\}$

$\{2,3\} \not\subseteq \{1,2\}$

O1 is not satisfied.

Example $(\mathbb{R}, <)$ usual "less than") etc.

Orders

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Examples

- Given X a set with more than 1 element, then $\mathcal{P}(X)$ with $\subset$ is not an ordered set.
- The number sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, with the usual $<$ are ordered sets

Bounded sets

Let $(S, <)$ be an ordered set.

Bounded sets

$s \leq x$

$s < x$ or $s = x$

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Let $(S, <)$ be an ordered set.
A subset $E \subseteq S$ is said to be bounded above (below) if there is an $s \in S$ such that $s \geq x$ $(s \leq x)$ for all $x \in E$. In this case we say that s is an upper (lower) bound for E.

If E is both bounded above and below we say that it is bounded.

Example $(\mathbb{Q}, <)$ $E = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$

bounded above $x \in E \Rightarrow x < 2$

bounded below $x \in E \Rightarrow x > -2$

$\Rightarrow$ This is Bounded

2 is a upper bound for E $\rightarrow$ there are many

-2 is a lower bound for E

Bounded sets

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If E is both bounded above and below we say that it is bounded.

Definition

Let $E \subseteq S$ bounded above (below) a supremum (infimum) for E is $\alpha \in S$ such that

1. α is an upper (lower) bound for E;
2. every upper (lower) bound for E, γ satisfies $\gamma \geq \alpha$ ($\gamma \leq \alpha$).

The E from before has no supremum in $\mathbb{Q}$ and no infimum.

Example $(\mathbb{R}, <)$

$$E = \{ x \in \mathbb{R} \ / \ x < 1 \}$$

This has a sup

$$\sup_{\mathbb{R}} E = 1$$

$1 \in \mathbb{R}$ $\quad$ $1 > x$ for all $x \in \mathbb{R}$

$\boxed{\gamma > x}$ for all $x \in \mathbb{R} \Rightarrow \gamma \geq 1$

Suppose otherwise $\gamma < 1$

$\gamma + \frac{\gamma + 1}{2}$ should be the mid point of γ and 1

$\swarrow x \in E$ $\quad$ $x > \gamma$

$\Longrightarrow \gamma$ is not an UP

Bounded sets

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2. every upper (lower) bound for E, γ satisfies $\gamma \geq \alpha$ ($\gamma \leq \alpha$).

If such α exists it is clearly unique and we denote it by

$$\alpha = \sup_S(E) \quad (\inf_S(E))$$

Least upper bound property

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We say that an ordered set $(S, <)$ has the least upper bound property (LUP) if every non empty subset E, bounded above has a supremum

Least upper bound property

We say that an ordered set $(S, <)$ has the least upper bound property (LUP) if every non empty subset E, bounded above has a supremum

We say that an ordered set $(S, <)$ has the greatest lower bound property (GLP) if every non empty subset E, bounded below has a infimum

Examples

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tho sup should be $\sqrt{2} \notin \mathbb{Q}$

- $\mathbb{Q}$ has not the LUP since $E = \{x | x^2 < 2\}$ is nonempty, bounded above but has no sup.
- Is a consequence of the well ordering of the integers that $\mathbb{Z}$ has the LUP. In addition we have that $\sup(E) \in E$.

$\phi \neq E \subseteq \mathbb{Z}$ bounded below (above)

$\Rightarrow E$ has a min (max)

a minimum (maximum) for $E \subseteq (\mathbb{S}, <)$ is an $\alpha \in E$

such that $\alpha \leq x$ ($\alpha \geq x$) for all $x \in E$

(a lower (upper) bound that belongs to the set)

Equivalence of LUP and GLP

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Theorem

If $(S, <)$ has the LUP then it has the GLP

Proof / S LUP $E \subseteq S$ bounded above HP
$\Longrightarrow$ S has a sup.

$\emptyset \neq B \subseteq S$ bounded below want to show that it has an inf

$L = \{ x \in S \mid x \leq b \text{ for all } b \text{ in } B \} \subseteq S$
$= \{ x \in S \mid x \text{ is a lower bound for } B \} \neq \emptyset$

L ($\neq \emptyset$) is bounded above: take $b \in B$ ($\neq \emptyset$)
by def of L we have that

$b \geq x$ for all $x \in L$ b is an upper bound

L has a sup $\sup_S L =: \alpha \in S$

Claim $\alpha = \inf_S B$

Need   1) $\alpha \in S$ ✓   an lower bound for B

2) any other lower bound for B is $\leq \alpha$

prop 2 of being a sup

1) let $x < \alpha$ then $x \notin B$ all the elements of B are upper bound for L
all the elements of B are $\geq \alpha$.

If $\beta > \alpha = \sup_S L$

$L = \{ \text{lower bounds for } B \}$

$\beta \notin L$ $\qquad$ $\beta \in L$ $\qquad$ $\beta \in L$

$\qquad\qquad\qquad\qquad\qquad\qquad \geq \sup_S L$

$\Rightarrow$ β is not a lower bound for B

<u>Contrapositive</u> $\quad$ β is a lower bound for B

$\Rightarrow$ $\beta \leq \alpha$

Questions?

Section 3
Fields

Recall the definition of field

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A *field* is a set F equipped with two binary operations, addition $(+)$ and multiplication $(\cdot)$, satisfying the following properties for all elements $a, b, c \in F$: Prototype $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

1. **Closure under Addition:** $a + b \in F$.
2. **Associativity of Addition:** $(a + b) + c = a + (b + c)$.
3. **Existence of an Additive Identity:** There exists an element $0 \in F$ such that $a + 0 = a$ for all $a \in F$.
4. **Existence of Additive Inverses:** For each $a \in F$, there exists an element $(-a) \in F$ such that $a + (-a) = 0$.
5. **Closure under Multiplication:** $a \cdot b \in F$.
6. **Associativity of Multiplication:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
7. **Existence of a Multiplicative Identity:** There exists an element $1 \in F$ such that $a \cdot 1 = a$ for all $a \in F$, where $1 \neq 0$.
8. **Existence of Multiplicative Inverses:** For each $a \in F$ such that $a \neq 0$, there exists an element $a^{-1} \in F$ such that $a \cdot a^{-1} = 1$.
9. **Distributive Property:** $a \cdot (b + c) = a \cdot b + a \cdot c$.

Ordered fields

An ordered field is a field F with and order $<$ such that

OF1 $x + y < x + z$ for all x, y, z in F with $y < z$.

OF2 $xy > 0$ for all x ad y in F with $x > 0$.

If $x > 0$ we say that it is positive. If $x < 0$ we say it is negative.

Examples $\mathbb{Q}$, $\mathbb{R}$

NO LUP

No example $\mathbb{C}$ we cannot decide

you cannot give an order on $\mathbb{C}$ compatible with $<$ on $\mathbb{R}$ & the multiplication

if $i > 0$ or $i < 0$

$-1 = i \cdot i > 0$ (OF2)

$i < 0$ $-i > 0$

$-1 = (-i)(-i) > 0$

Ordered fields

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OF1 $x + y < x + z$ for all x, y, z in F with $y < z$.

OF2 $xy > 0$ for all x ad y in F with $x > 0$ and $y > 0$.

If $x > 0$ we say that it is positive. If $x < 0$ we say it is negative.

Proposition

The following are true in an ordered field $(F, <)$. $1 \in F \rightsquigarrow -1 \in F$

1. if $x > 0$ then $-1 \cdot x = -x < 0$
2. if $x > 0$ and $y < z$, then $xy < xz$. ·
3. if $x < 0$ and $y < z$, then $xy > xz$. ·
4. if $x \neq 0$ then $x^2 > 0$. In particular $1 > 0$. $1 = (1 \cdot 1)$
5. if $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.

Not hard $\rightsquigarrow$ if you have problems just write in the forum.

Example

The field of comples numbers is not an ordered field.

The Real Numbers

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Theorem

There is a unique ordered field with the LUP $\mathbb{R}$ that contains $\mathbb{Q}$ as a sub(ordered)field. (NO PROOF)

The Real Numbers

Theorem

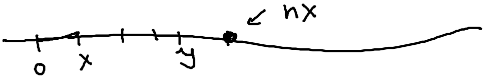
There is a unique ordered field with the LUP $\mathbb{R}$ that contains $\mathbb{Q}$ as a sub(ordered)field.

The archimedian property

If x and y are real numbers with $x > 0$, then there is $n \in \mathbb{Z}$, $n > 0$, such that $nx > y$.

Density of $\mathbb{Q}$

Given two real numbers $x < y$ then there is a rational number q such that $x < q < y$.



Proof (Archimedean) $x > 0$

if $y \leq 0$ $y < x$ take $n = 1 \in \mathbb{Z}^+$

We can assume that $y > 0$

$\underline{\phi \neq} A = \{ nx \mid n \in \mathbb{Z} \ \ n > 0 \} \ni x = 1 \cdot x$

Suppose by contradiction

$y \geq nx$ for all $n \in \mathbb{Z}^+$ y is an UP

for A (bounded above) $\exists$ $\sup_{\mathbb{R}} A = \alpha$

$\alpha \geq x^{\in A}$ $x > 0$ $\alpha - x < \alpha$

$\alpha - x$ is not an upper bound

$\exists\, m \cdot x > \alpha - x$, $m \in A$

$\Rightarrow (m+1) \cdot x > \alpha$

α is not an upper bound. ☺

Proof (Density)

$x, y \in \mathbb{R}$ $\quad x < y$ $\quad y - x > 0$

Archimedean $\exists\, n \in \mathbb{Z}^+$ $\quad n(y-x) > 1$

$ny > 1 + nx$ $\quad 1 > -nx$

Archimed $\quad k \quad k \cdot 1 > nx$

$-1 < nx < k$

$\emptyset \neq A = \{ h \in \mathbb{Z} \mid h < nx \} \ni -1$

bounded above by $\underline{k}$.

It has a max (well ordering) $\bar{h}$

$$x < \frac{\bar{h}+1}{n} < y$$

$$\in \mathbb{Q}$$

$$\bar{h} < nx < \bar{h}+1 \qquad x < \frac{\bar{h}+1}{n}$$

$$< nx+1 < ny$$

Questions?

The complex numbers

The field of the complex number $\mathbb{C}$ consist of pairs of real numbers (a, b) with the following binary operations

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc)$$

The zero is given by the number (0,0) and the multiplicative identity is given by $(1, 0)$.

The complex numbers

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The zero is given by the number (0,0) and the multiplicative identity is given by $(1, 0)$. One can check that

$i^2 = (-1, 0)$

$$(a, b) = (a, 0) + (0, 1)(b, 0),$$

Thus if we set $i = (0, 1)$ and we identify $\{(a, 0) | a \in \mathbb{R}\}$ with the real numbers $\mathbb{R}$ we can write

$$\boxed{(a, b) = a + ib,}$$

Given $z = a + ib$ a complex number, the real number a is called the real part of z and it is denoted by $\Re(z)$, while $b = \Im(z)$ is the imaginary part of z.

! Not ib.

The extended Real numbers

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The *extended real numbers* are represented by the set

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$$

NOT a field

Addition and Multiplication: The addition and multiplication can be partially extended to the extended real numbers in teh the

$$a + (+\infty) = +\infty \quad \text{for any } a \in \mathbb{R}$$

$$a + (-\infty) = -\infty \quad \text{for any } a \in \mathbb{R}$$

$$a \cdot (\pm\infty) = \begin{cases} \pm\infty & \text{if } a > 0 \\ \mp\infty & \text{if } a < 0 \end{cases}$$

$$a/(\pm\infty) = 0 \text{ if } a \neq 0$$

Undefined Cases:

- The sum $+\infty + (-\infty)$ is undefined.
- The product $0 \cdot (+\infty)$, $0 \cdot (-\infty)$, and $(+\infty) \cdot (-\infty)$ are undefined.

Questions?

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Section 4
Cardinality

Cardinality

Given two sets A and B we say that they have the same cardinality if there is a bijective function $f : A \to B$.

(the fact that this is well defined is a consequence of the axiom of choice)

Cardinality

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Given two sets A and B we say that they have the same cardinality if there is a bijective function $f : A \to B$. This yields an equivalence relation $\sim$.

Cardinality

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$\to J_0 = \emptyset$

Given two sets A and B we say that they have the same cardinality if there is a bijective function $f : A \to B$. This yields an equivalence relation $\sim$.Consider the sets $J_n := \{1, 2, \ldots, n\}$ and $J = \mathbb{Z}^+$

Definition

Given a set A we say that

- A is finite if $A \sim J_n$ for some n. In this case we have that n is unique and we set $|A| = n$. → the same # of element.
- A is infinite if it is not finite.
- A is countable if $A \sim J$
- A is uncountable if it is neither finite nor countable
- A is at most countable if it not uncountable.

Sequences

A sequence on a set A (or with values in A) is a function

$\mathbb{Z}_{\geq 0}$

$$f : J \to A$$

Usually, the n-th term of the sequence $f(n)$ is denoted by a_n or x_n.

p_n

Sequences

A sequence on a set A (or with values in A) is a function

$$f : J \to A$$

Usually, the n-th term of the sequence $f(n)$ is denoted by a_n or x_n.

If A is a countable set then there is a bijective correspondence $f : J \to A$ so we can write

$$A = \{f(n) | n \in J\}$$

In particular countable sets can be rearranged in sequences.

Unions of (at most) countable sets

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Proposition

A countable union of countable sets is countable

Corollary

An at most countable union of at most countable sets is at most countable.

Proof (Prop) $\{E_n\}_{n\in\mathbb{Z}^+}$ a (countable) collection of countable sets

$E_n = \{ x_{1,n} \; x_{2,n} \; \dots \}$

↳ countable.

$E_1 \subseteq \bigcup_{n=1}^{\infty} E_n = \{x_{i,j}\}_{i,j\in\mathbb{Z}^+}$ is not finite

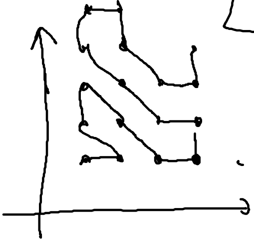
We need to show that is contained into countable se.

$$g: \mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \bigcup E_n$$
$$(i,j) \longmapsto x_{ij}$$

it might not be injective

We have $Z \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ such that is bijective

↳ Countable ⟹ :)



✓

$\mathbb{Q}$ is countable

$$J \subseteq \mathbb{Q}^+ = \left\{ \frac{m}{n} \;\middle|\; m,n > 0 \right\} \longleftarrow \mathbb{Z}^+ \times \mathbb{Z}^+$$

$$\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$$

$\hookrightarrow$ contabile

$$- \; : \; \mathbb{Q}^+ \longrightarrow \mathbb{Q}^- \qquad \text{biject.}$$

$$x \longmapsto -x$$

Products of countable sets

Recall that given a set A, the set

$$A^n := \{(a_1, \ldots, a_n) | a_i \in A\}$$

Proposition

If A is countable then A^n is countable

Corollary

If $\mathbb{Q}$ is countable.

An uncountable set

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Theorem

Let A be the set of sequences with values in $\{0, 1\}$. Then, A is not countable.

As a consequence we have that the set of real numbers is not countable.

Proof $E \subseteq A$ at most countable $\Longrightarrow E \neq A$

$E = \{ s_1, s_2, s_3 \dots \}$ I create a new sequence $s \in A$ such that $s \notin E$

$s : \mathbb{Z}^+ \longrightarrow A$

$s(i) = \begin{cases} 0 & \text{if } i\text{th digit of } s_i \text{ is } 1 \\ 1 & \text{—— is } 0 \end{cases}$

$s \neq s_j$ for all $s_j \in E$

$s = s_j$ they have to assume the same value

$s(j) \neq s_j(j)$

$s \notin E$ $s \in A$

Corollary $\Rightarrow$ $[0,1]$ is not countable

$\mathbb{R}$ not countable

$x \in \mathbb{R} \Rightarrow x =$ sequences in $\{0,1\}$

$\mathbb{R} \hookrightarrow$ Sequences of $\{0,1\}$

$[0,1] \Leftrightarrow$ sequences in $\{0,1\}$

Thank you for your attention!