Introduction to Real Analysis

Lecture 2: Metric spaces

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.

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Lecture Plan

- Metric spaces (Rudin 2.15-2.30)
- Compact sets I (Rudin 1.31-1.33)

Section 1
Metric spaces

Definition

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A metric space is a pair (X, d), where X is a set, whose element we will call points, and $d : X \times X \to \mathbb{R}$ is a function, called distance, satisfying the following properties for all $x, y, z \in X$:

1. **Non-negativity:** $d(x, y) \geq 0$.
2. **Identity of indiscernibles:** $d(x, y) = 0$ if and only if $x = y$.
3. **Symmetry:** $d(x, y) = d(y, x)$.
4. **Triangle inequality:** $d(x, z) \leq d(x, y) + d(y, z)$.

Examples

- The Euclidean distance

$$d(\mathbf{x},\mathbf{y}) = \left(\sum_{j=\cancel{0}\,1}^{n}(x_j - y_j)^2\right)^{\frac{1}{2}}$$

  gives a distance in $\mathbb{R}^n$.
- The sup distance

$$d(\mathbf{x},\mathbf{y}) = \max_j \{|x_j - y_j|\}$$

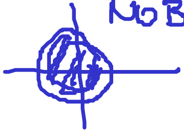
  gives a distance in $\mathbb{R}^n$.
- If (X, d) is a metric space and $Y \subseteq X$, then restricting d to $Y \times Y$ yields a distance on Y.

Exercise: show that $d|_{Y \times Y}$ gives you a distance

Vocabulary

$\mathbb{R}^2$

$r = 1$



NO BORDER!

Let (X, d) be a metric space.

- Given $p \in X$ and $r \in \mathbb{R}^+$, the neighbourhood of p with radius r is

$$N_r(p) = \{x \in X | d(x, p) < r\}$$

Remark

Observe that if $Y \subseteq X$ and p in Y, the neighbourhood of radius r of p in Y is

$$N_r(p) \cap Y$$

Vocabulary

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Remark

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$$N_r(p) \cap Y$$

- Given $E \subseteq X$, $p \in X$ is a limit point for E if, for every $r > 0$ we have that

$$(N_r(p) \cap E) \setminus \{p\} \neq \emptyset,$$

Exer

We denote by E' the set of limits points of E.

- A point $p \in E$ which is not a limit point is called an isolated point of E.

$E = (0, \tfrac{1}{2}) \cup \{1\}$ isolated 1

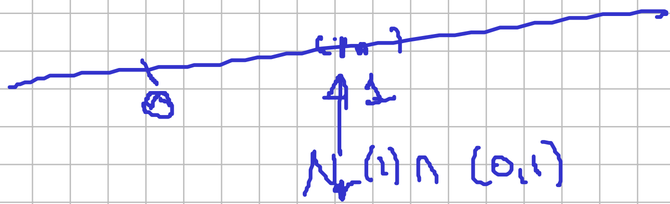
$(N_{1/3}(1) \cap E) \setminus \{1\} = \phi$

Example

$X = \mathbb{R}$ $d(x,y) = |x-y|$

$E = (0,1)$

1 is a limit point



Vocabulary II

(0,1) NOT CLOSED

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Let (X, d) be a metric space.

- $E \subseteq X$ is closed if contains all its limit points.
- Given $E \subseteq X$, $p \in X$ is a interior point for E if, there is an $r > 0$ such that $N_r(p) \subseteq E$ We denote by $\mathring{E}$ the set of limits points of E.
- E is open if $E = \mathring{E}$.
- E is perfect if ti is closed and every point of E is a limit point of E.
- $E \subseteq X$ is bounded if $E \subseteq N_r(p)$ for some $p \in X$ and $r > 0$.
- E is dense in X if every point of X is a limit point of E.

→ closed + no isolated pt

Exp $(0,1) \subseteq N_2(0)$ is bounded

Example

$(0,1)$ $\quad p = \frac{1}{2}$ interior.

$N_{\frac{1}{3}}(\frac{1}{2}) \subseteq (0,1)$

$(0,1)$ $\quad \frac{1}{2}$ $\quad N_r(\frac{1}{2}) = (1/2 - r, \frac{1}{2} + r)$

$r > 0$ $\quad \left(r(0,1) \cap N_r(1/2) \right) \setminus \{\frac{1}{2}\} \neq \emptyset$ $\quad$ interval.

Open neighbourhood

Theorem

Every neighbourhood is open.

Proof (X, d)

$p \in X$ $\quad r > 0$ $\quad r \in \mathbb{R}$ $\quad \underline{r - d(q,p)}$;

$N_r(p)$ $\quad q \in N_r(p) \Rightarrow$

Need $\bar{r}$ $\quad N_{\bar{r}}(q) \subseteq N_r(p)$

$\bar{r} = \rho/3$ does the deal

$x \in N_{\frac{r}{3}}(a)$

$d(x,a) < \rho/3$

ρ

$r_x + \varepsilon < r$

$\varepsilon < r - \rho$

$d(x,p) \leq d(x,a) + d(a,p)$

$\triangle$ ε

$\leq \frac{\rho}{3} + \rho$

$= \frac{\rho}{3} + r - d(x,y) < r$

Many points close to a limit point

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Theorem

If $p \in E$ is a limit point, then for every neighbourhood N of p we have that $N \cap E$ is infinite.

Corollary

Finite sets do not have limit points.

p $\quad N = N_r(p) \quad n \in \mathbb{Z}^+$

$N_r(p) \cap E \quad N_{r/n}(p) \cap E \neq \emptyset$

Open and closed

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Theorem

A set E is open if, and only if, its complementary E^c is closed.

Corollary

A set E is closed if, and only if, E^c is open.

E open $\quad N_r(p) \subseteq E$ for all $p \in E$

and some $r > 0$

E^c closed 

E^c contains all the limit pts

$p \in X \quad p \notin E^c \quad (p \in E)$

$\Rightarrow \exists\, r>0 \quad N_r(p) \subseteq E$

$(N_r(p) \cap E^c) \setminus \{p\} = \emptyset$

($\Leftarrow$) E^c is closed E^c contains all the limits pts $p \in E$ is not a limit pt for E^c

$\exists\ \underline{r>0}\ \big(\underbrace{N_r(p)\cap E^c}_{\subseteq E^c}\big) \setminus \underbrace{\{p\}}_{\text{not rec.}} = \emptyset$

$N_r(p) \cap E^c = \emptyset$

$\boxed{N_r(p) \subseteq E}$

$\Rightarrow p \in \mathring{E}$

Operations on open and closed sets

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Theorem

1. Unions of open sets are open
2. Intersections of closed sets are closed
3. Finite intersection of open sets are open
4. Finite union of closed sets are closed.

Proof X $\{U_\alpha\}$ $\alpha \in A$

U_α open $\bigcup_{\alpha \in A} U_\alpha$ open.

$\underline{p \in \bigcup_{\alpha \in A} U_\alpha}$ need interior

$p \in \underset{\text{open}}{U_{\tilde{\alpha}}}$ for some $\tilde{\alpha} \in A$

$\underbrace{\exists\, r > 0 \quad N_r(p) \subseteq U_{\tilde{\alpha}} \subseteq \bigcup_{\alpha \in A} U_\alpha}$ OPEN

3) $U_1 \dots U_n$ open

$\underline{p \in \bigcap^n U_i}$ $\quad p \in \underset{\text{open}}{U_i}$ for all i

for every i $\exists\, r_i$ $\quad N_{r_i}(p) \subseteq U_i$

$\boxed{\rho} = \min\{r_1, r_2, \dots r_n\} \in \mathbb{R}^+$

$N_\rho(p) \subseteq U_i$ for all i $\Rightarrow \boxed{N_\rho(p)} \subseteq \bigcap^n U_i$

$N_\rho(p) \subseteq N_{r_i}(p) \subseteq U_i$

Operations on open and closed sets

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Theorem

1. Unions of open sets are open
2. Intersections of closed sets are closed
3. Finite intersection of open sets are open
4. Finite union of closed sets are closed.

Example

The intersection of open intervals of the form $(-\frac{1}{n}, \frac{1}{n})$ is $\{0\}$ which is not open.

$\{0\} = \bigcap (-1/n, 1/n)$ $\quad (-r, r)$

$x > 0 \quad \exists N \quad \frac{1}{N} < x \quad x \notin (-1/N, 1/N)$

similarly $x < 0$

Closure

Given $E \subset X$, its closure is

$$\overline{E} = E \cup E'$$

Theorem

1. $\overline{E}$ is closed
2. E is closed if, and only if, $E = \overline{E}$.
3. If C is closed and $C \supseteq E$, then $C \supseteq \overline{E}$. That is $\overline{E}$ is the smallest closed set containing E.

Proof: $p \in \bar{E} = E \cup E'$

want $p \in \bar{E}'$

$\bar{E}$ closed $\iff$ $\bar{E}^c$ open

$p \in \bar{E}^c$ $\quad p \notin E$ $\quad p \notin E'$

$\exists r > 0$ such ~~that~~ $\left(N_r(p) \cap E\right) \setminus \{p\} = \emptyset$

$N_r(p) \cap E = \emptyset$ $\quad N_r(p) \subseteq \bar{E}^c$ open.

if $E = \bar{E}$ closed

We have to show ~~that~~

if E is closed $E = \bar{E}$

$E \subseteq \bar{E} = E \cup E'$

WANT $\bar{E} \subseteq E$

$p \in \bar{E}$ $\quad p \in E$ ok

$p \notin E$ then $p \in E' \subseteq E$

no matter where p sits $p \in E$

C closed $\quad C \supseteq \bar{E}$

$C \overset{?}{\supseteq} E \cup E'$

Need $E' \subseteq C$

$p \in E'$ $\forall r$ $\left(N_r(p) \cap E\right) \setminus \{p\} \neq \emptyset$

$\left(N_r(p) \cap C\right) \setminus \{p\}$ $\quad p \in E^{c\prime}$

$E' \subseteq C$!! $\quad p \in E' \subseteq C' \subseteq C$ $\quad C$ closed

Limit and sup

$\mathbb{R} = X \quad d(x,y) = |x-y| \quad \overline{E} \supseteq E$

Theorem

Let E be a non-empty subset of $\mathbb{R}$, bounded above. Then $\sup(E) \in \overline{E}$.

if E is closed $\sup(E) \in E$

if $\underline{\sup E \in E}$ ok. Assume $\sup(E) \notin E$

$\underline{\text{Proof}}$: $\underline{\varepsilon > 0}$ $\quad \sup(E) - \varepsilon < \sup(E)$

not an upper bound

$\exists \; x \in E \quad \sup(E) - \varepsilon < x < \sup(E)$

$(N_\varepsilon(\sup(E))) \cap E \setminus \{\sup E\} \neq \emptyset \Rightarrow \sup(E) \in E'$

$\overline{E}$

Relatively opens sets

If $Y \subseteq X$ and $E \subseteq Y$ we could have that E is open in Y (relatively to Y) but not in X

Ex

$[0,1)$ is not open in $\mathbb{R}$

$Y = [0,1]$ is open in $[0,1]$

$N_r^Y(0) = N_r^{\mathbb{R}}(p) \cap Y = Y$

Relatively opens sets

If $Y \subseteq X$ and $E \subseteq Y$ we could have that E is open in Y (relatively to Y) but not in X

Theorem

A set $E \subseteq Y$ is open relatively to Y if, and only if, there is an open set of X, U with $E = Y \cap U$.

$E = Y \cap U$ $\quad p \in E \quad p \in U$

$\exists r > 0$

$N_r^X(p) \subseteq U$

$N_r^Y(p) = N_r^X(p) \cap Y \subseteq U \cap Y \subseteq E$

$E \subset Y$ open (with r.t $d|_{Y \times Y}$)

$p \in E \quad \exists\, r_p \quad N^Y_{r_p}(p) \subseteq E$

OPEN

$$\underbrace{\Bigg(\underbrace{\bigcup_{p \in E} N^X_{r_p}(p)}_{\cup}\Bigg) \cap Y}_{\shortparallel} = E$$

$$E \subseteq \bigcup_{p \in E} N^Y_{r_p}(p) \subseteq E$$

Questions?

Section 2
Compact sets and relatively compact sets

Open covering

Given $E \subseteq X$, an open covering of E is a collection $\{U_\alpha\}_{\alpha \in A}$ of open sets of X such that

$$E \subseteq \bigcup_{\alpha} U_\alpha$$

Compact

Given $K \subseteq X$, we say that it is compact if, for every open covering $\{U_\alpha\}_{\alpha \in A}$ there are finitely many $\alpha_1, \ldots, \alpha_n$ such that

$$E \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$

Relatively compact sets

Theorem

If $K \subseteq Y \subseteq X$ we have that K is compact realtively to Y if, and only if, it is compact in X.

Proof

K cpt in X

$\{U_\alpha\}$ an open cover of K in Y

$\boxed{U_\alpha = V_\alpha \cap Y}$ V_α open in X

$K \subseteq \bigcup_\alpha U_\alpha = \bigcup_\alpha (V_\alpha \cap Y) \subseteq \bigcup_\alpha V_\alpha$

K cpt in X

$\exists\ \alpha_1 \dots \alpha_n$

$K \subseteq \bigcup^n V_{\alpha_i}$ $\quad K \subseteq Y$

$K = K \cap Y \subseteq \bigcup_i^n (V_{\alpha_i} \cap Y)$

$= \bigcup_i^n V_{\alpha_i}$

K compact in Y

$\underline{V_\alpha}$ open cover in $\underline{X}$

$K \subseteq \bigcup V_\alpha$ open cover

$K = K \cap Y \subseteq \bigcup_\alpha (V_\alpha \cap Y)$

$\exists\ \alpha_1 \dots \alpha_n$

$K \subseteq \bigcup_\alpha (V_{\alpha_i} \cap Y) \subseteq \boxed{\bigcup_i V_{\alpha_i}}$

Questions?

Thank you for your attention!