MM5023
Lecture 1

Logistic:

Schedule: Mondays- Thursday 9 -11, with TA session 11-12.
There might be small variation soon look at the official schedule

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Book: Grimaldi 's Introduction to discrete and combinatorial mathematics

Bonus points:

Lecture notes:

Exam: January 12th 2024. You have to be here!

Course Plan:

- Review and counting (inclusion/exclusion)
- Rook polynomial
- Generating functions
- Graph theory (coloring, Hamilton paths, tree, weighted trees, optimization, max flow min cut)
- Finite geometry and Latin squares

Today:

- Review: set functions equivalence relations
- Review: principles of counting (counting using functions)
- Pigeon holes principle and generalized pigeon hole principle.

What is a set?

Set Notation

Stockholm University

A set is a collection of elements.

Example

$$
\begin{aligned}
P &= \{\text{red, yellow, blue}\} \\
  &= \{\text{primary colors}\} \\
  &= \{c \mid c \text{ is a primary color}\}
\end{aligned}
$$

"such that"

blue $\in P$ purple $\notin P$

The symbol $\emptyset$ denotes the set with no element, aka the empty set.

Example

$$A = \{ \{1, 2\}, \{3\}, 2, 4, \{1, 5\} \}$$

$3 \notin A$ $\qquad\qquad$ $\{3\} \in A$

$1 \notin A$

$\{1\} \notin A$ $\qquad\qquad$ $\{1, 2\} \in A$

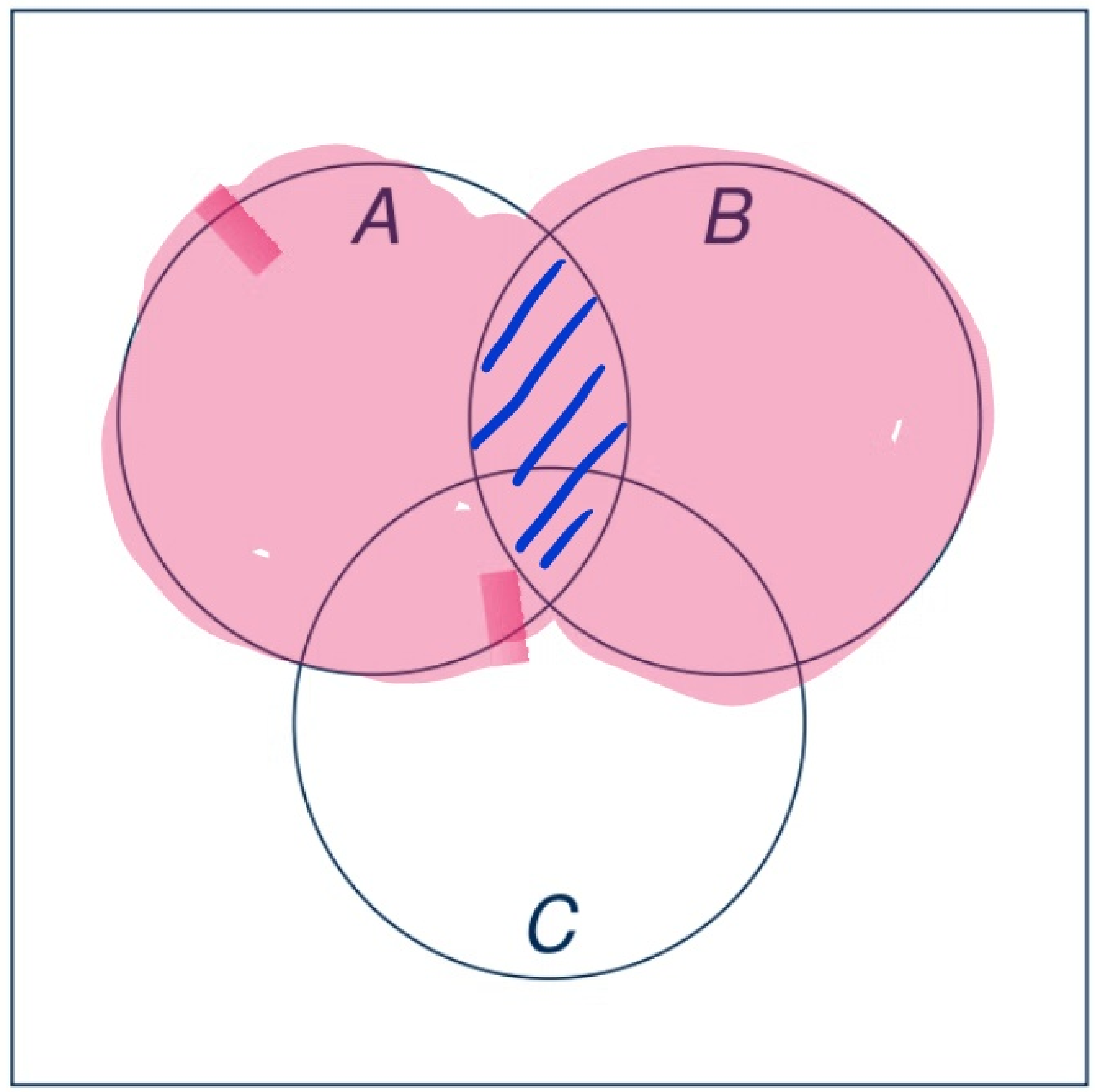
Operations on sets

In order to leave barbers alone we are always going to assume that all our subset are subsets of a universe set.

Union

Given two sets A and B, their union is

$$A \cup B := \{x \mid x \in A, \text{OR } x \in B\}$$



$A \cap B$

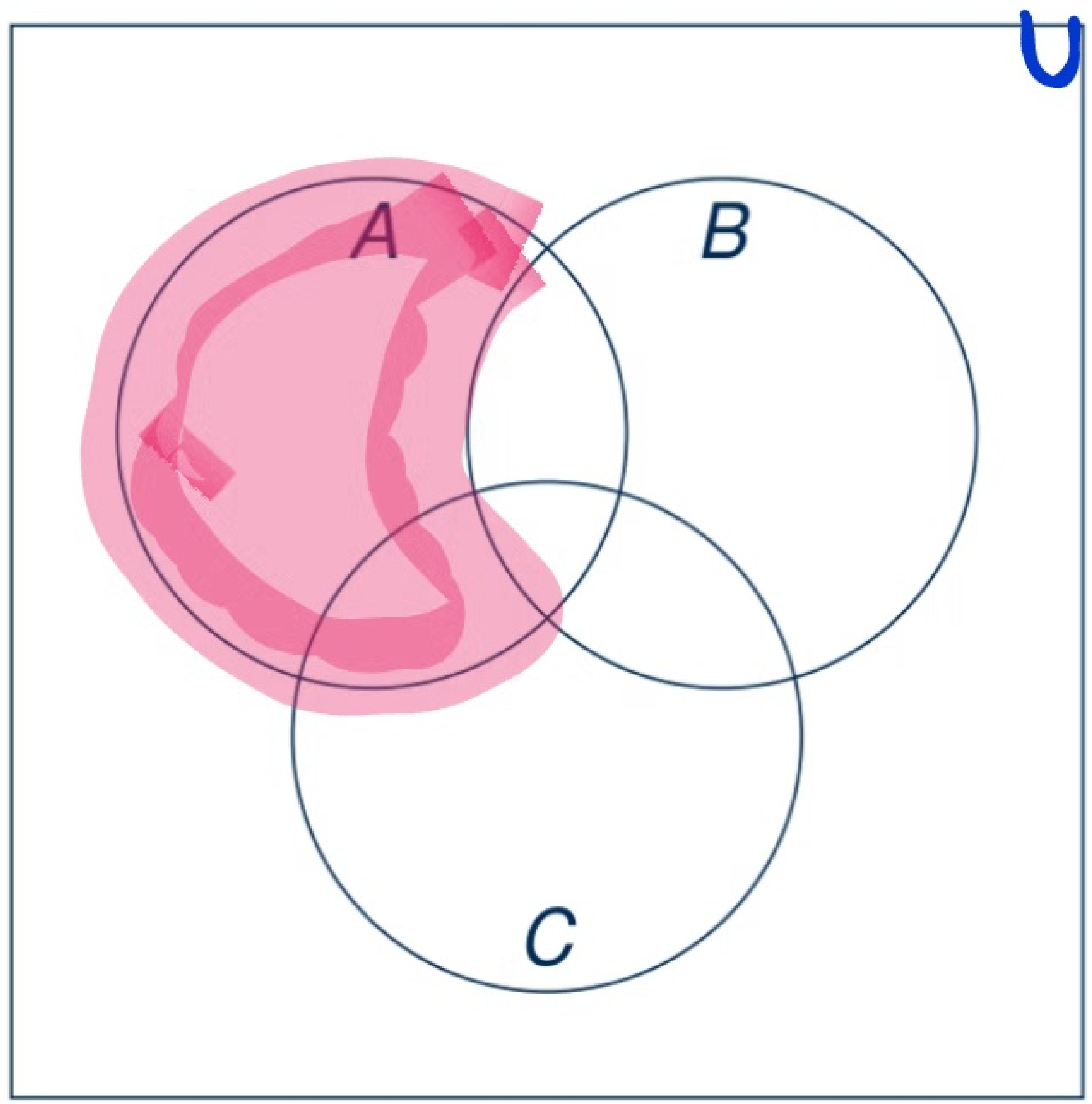
$= \{x \in A \text{ AND } x \in B\}$

INTERSECTION

Excision

Given two sets A and B, we have that

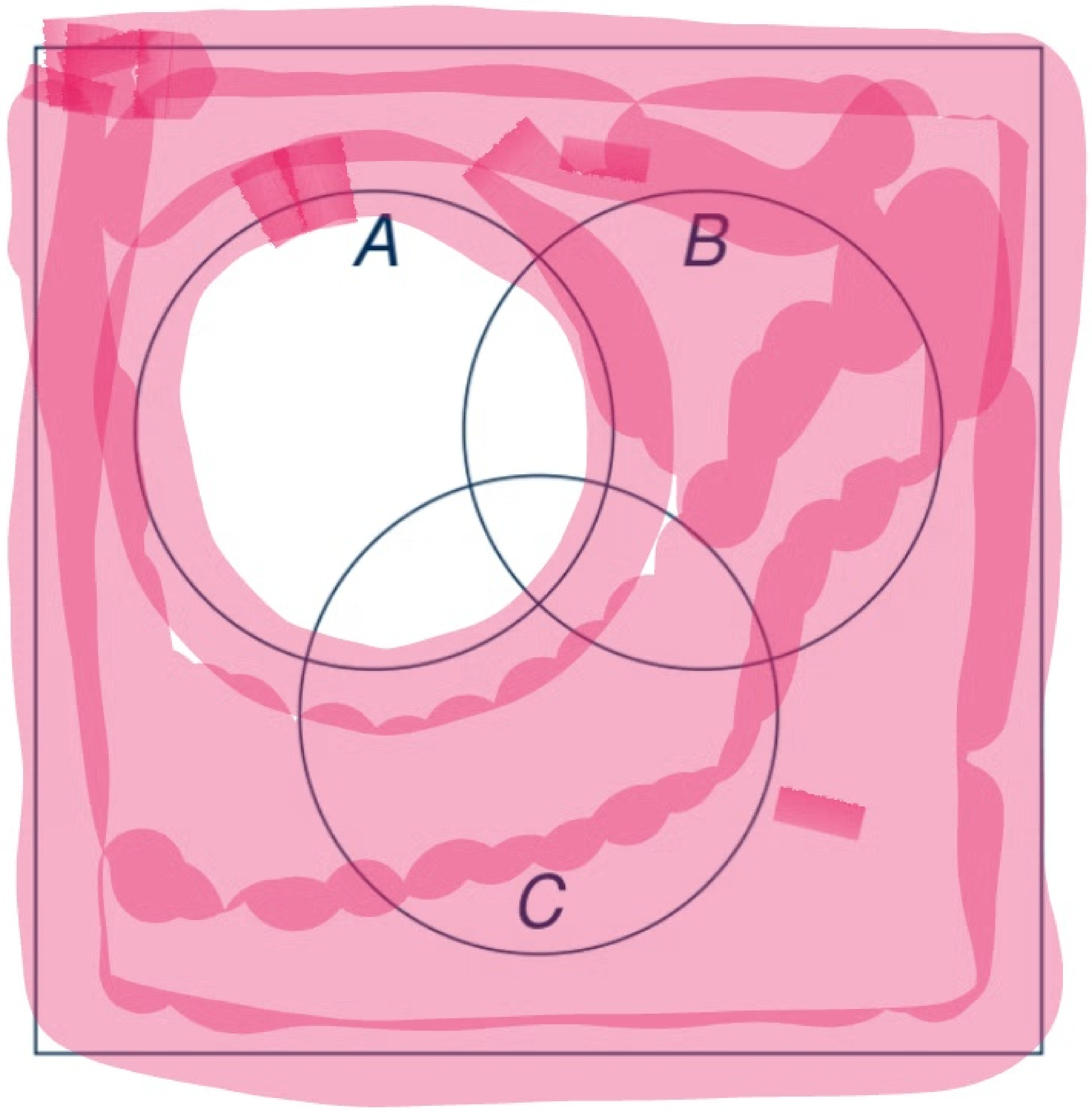
$$A \backslash B := \{x \mid x \in A, \text{AND } x \notin B\}$$



Complementary

Given a sets A (living inside a universe set U), its complementary is

$$A^c := \{x \in U \mid x \notin A\}$$



Disjoint Union

A set A is the disjoint union of two sets B and C ($A \sqcup B$)

if

- $A = B \cup C$
- $B \cap C = \emptyset$

Example

$\mathbb{Z} = E \sqcup O$

$E = \{ x \in \mathbb{Z} \mid x \text{ is even} \}$

$O = \{ x \in \mathbb{Z} \mid x \text{ is odd} \}$

Cartesian product

A B two sets

$A \times B = \{ (x, y) \ / \ x \in A \ y \in B \}$

Order matters !

<u>Example</u> $A = \{1, 2, 3\}$ $B = \{1, 2\}$

$A \times B = \{ (1,1), (1,2), (2,1), (2,2), (3,1), (3,2) \}$

different elements !

What is a function f:A->B

A function is a correspondence assigning to <u>any</u> element $a \in A$ a <u>unique</u> element $b \in B$ denoted by $f(b)$

Relations

A, B sets

A relation from A to B is a subset

$$R \subseteq A \times B.$$

Example

$A = \{ \text{people} \}$ $\quad$ $B = \{ \text{country} \}$

$R = \{ (a, b) \mid a \text{ lives in } b \}$

$R = \{ (a, \text{Spain}) \mid a \in A \}$

aRb $\iff$ (a,b) is in R

Example $A = \{\text{people}\}$ $B = A$

$R = \{ (a, b) \ / \ a, \text{ and } b \text{ live in the same country} \}$

$a R a$

$a R b$ then $b R a$

$a R b$ $b R c \Rightarrow a R c$

} equivalance relation

$A = \{people\} = B$

$R = \{ (a,b) \mid a \text{ is older than } b \}$

aRb , bRc then aRc

A function $f: A \longrightarrow B$ is a relation $R \subseteq A \times B$ such that for every $a \in A$ there is a unique $b \in B$, denoted by $f(a)$ such that $a \, R \, f(a)$

<u>Example</u> People / Country vs function
Apple / Branche
$a R b \iff$ a was born in B.

$A = B = \mathbb{R}$

$x \mathcal{R} x^2 \quad \rightsquigarrow \quad f(x) = x^2$

$\leftarrow \mathcal{R} = \{ (x,y) \,/\, y = x^2 \}$

A function can be

1) injective $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ (1-1)

2) surjective Every $y \in B$ can be written as $f(x)$ with $x \in A$ (ONTO)

3) bijective if it is both

$$f : A \longrightarrow B$$

$C \subseteq A$ $\qquad f(C) = \{ y \in B \,/\, y = f(c) \;\; c \in C \}$

"image of C"

$D \subseteq B$

$f^{-1}(D) = \{ x \in A \,|\, f(x) \in D \}$ "pre image of D"

$D = \{b\}$ $\qquad f^{-1}(\{b\}) = f^{-1}(b)$

! f^{-1} is not the inverse !

$f: A \longrightarrow B$ bijective

then $\exists!$ $g: B \longrightarrow A$ such that

$f \circ g\,(b) = b$ for all $a \in A$

$g \circ f\,(a) = a$ for all $b \in B$

g is called the inverse of f and denoted by f^{-1}

SET

$f^{-1}(b) = f^{-1}(f(a)) = \{a\}$

FUNCTION

$f^{-1}(b) = a$

Counting

cardinality

Def: Let n be a natural number. We say that a set A has size n if there is a bijectve function

$$|A| = n$$

$$f: A \longrightarrow \underline{n} := \{1, 2, \ldots, n\}$$

Well defined $\rightsquigarrow$ **Pigeon hole principle**

Rule of product

If there are 2 tasks that can be performed in m ways & k ways, respectively, then there are m×k ways to perform both tasks

$$|A \times B| = |A| \cdot |B|$$

Rule of sum

If a single task can be performed in m ways and a second task can be performed in k ways & the 2 task cannot be performed at the same time then we have m+k ways to perform an action.

FORMAL

$$|A \cup B| = |A| + |B|$$

Permutations and combinations

$$n! \begin{cases} 1 & \text{if } n=0 \\ n(n-1)(n-2)\cdots 1 & \text{if } n>0 \end{cases}$$

A permutation of size r of n objects is an injective map

$$\underline{r} \longrightarrow \underline{n}$$

$P(n,r)$ = # permutations of size r of n object

$$= \frac{n!}{(n-r)!}$$

$f(1) \rightarrow n$ choices
$f(2) \rightarrow n-1$ choices
$\vdots$
$f(r) \rightarrow n-r+1$ choices

rule of product $= n(n-1)\cdots(n-r+1) = \dfrac{n!}{(n-r)!}$

<u>Combination</u> of size r of n objects is a subset $A \subseteq \underline{n}$ such that $|A| = r$

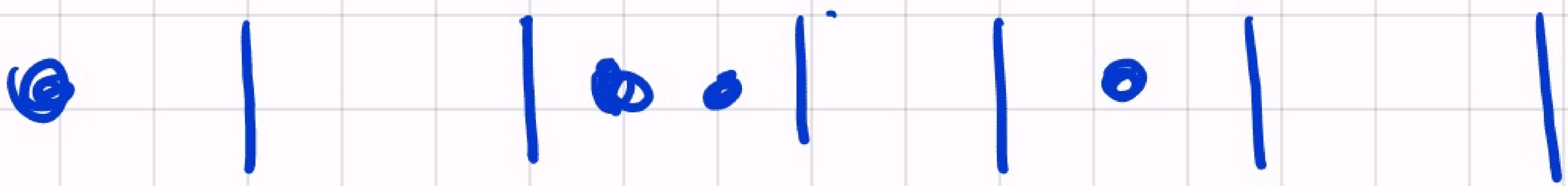
$$C(n,r) = \#\text{ combination} = \binom{n}{r}$$

$$= \frac{n!}{r!\,(n-r)!} = \frac{1}{r!}\,P(n,r)$$

Repetition \ ORDER	YES	NO
NO	Permutation $\frac{n!}{(n-r)!}$	Combination $\binom{n}{r}$
YES	$\binom{n!}{r_1 \ldots r_r}$	$\binom{n+r-1}{r}$

If you allow repetition & the order does not matter

$$\left(\frac{n+r-1}{r} \right)$$



Multinomial coefficient.

$$\frac{n!}{r_1! \cdots r_k!}$$

$$r_1 + \quad + r_k = r$$

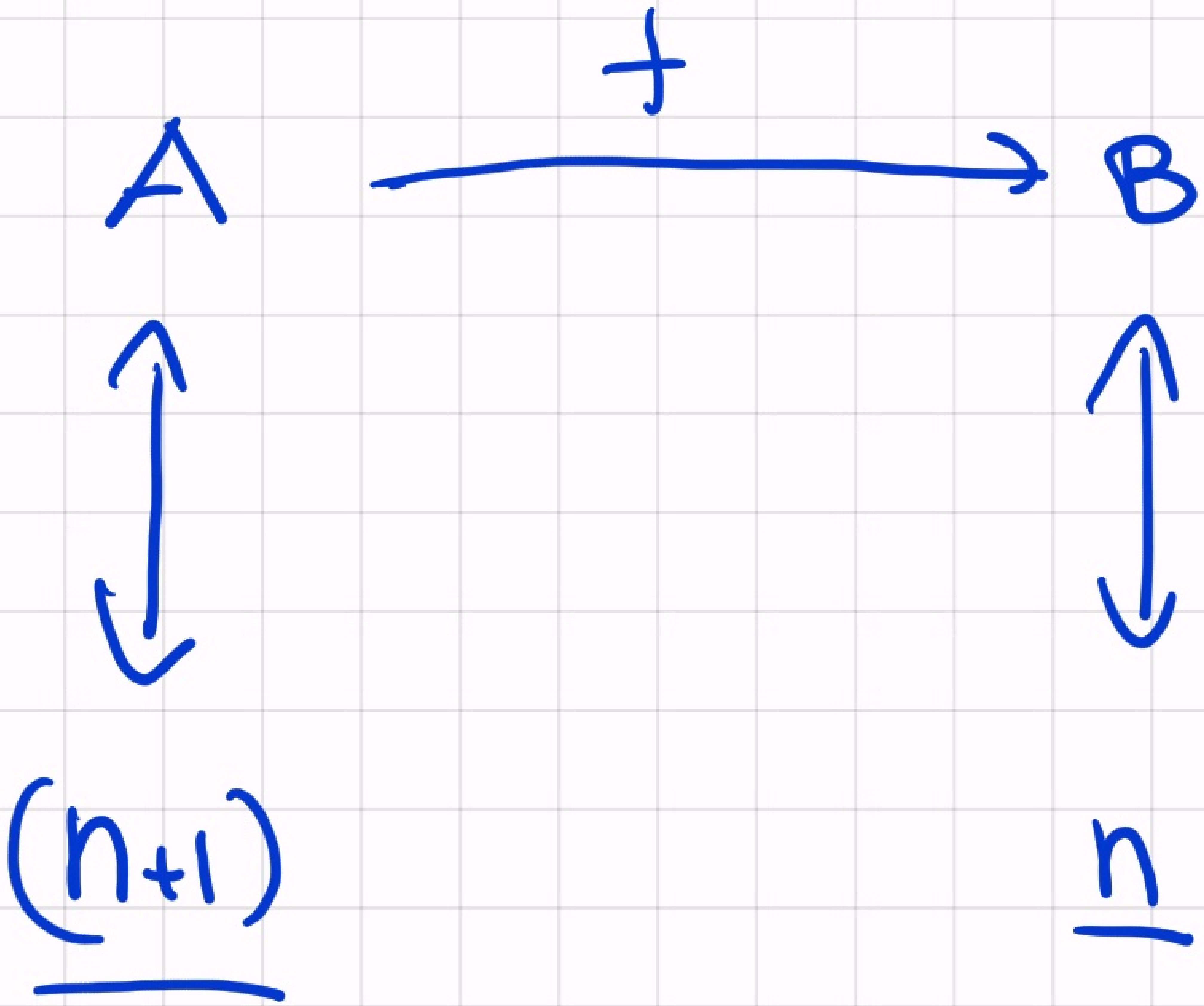
Pigeon holes principle

If there are $n+1$ pigeons & n nests then there is at least one nest with 2 pigeons

<u>Formal</u> if $m < k$ then there is not an injective map $f: k \longrightarrow m$.

$A = \{$ set of pigeons $\}$ $B = \{$ set of nest $\}$

$|A| = n+1$ $|B| = n$



$f(a)$ = nest of a

injective $\iff$ $|f^{-1}(b)| \leq 1$

by the principle f cannot be injective.
there is a $b \in B$ such that $f^{-1}(b)$ has size at least 2

Proof Uses the well ordering of the natural #

"Every $\neq \emptyset$ subset of the natural numbers has a min"

We reason by contradiction and assume that there are $m > k$ and an injective map

$$f : \underline{m} \to \underline{k}$$

We consider

$$S := \{ k \in \mathbb{N} \mid \exists\, m > k \text{ an } f : \underline{m} \to \underline{k} \text{ injective} \}$$

$\neq \emptyset \Rightarrow$ well ordering S has a min d

$d \in S$ so there is $m > d$ and

$f: \underline{m} \longrightarrow \underline{d}$ injective

I can assume $f(m) = d$

Now $f_{|(\underline{m-1})} : \underline{(m-1)} \longrightarrow \underline{(d-1)}$ injective

$m-1 > d-1$ since $m > d$

This contradict our choice of d !! ☺

→ Consequence : the size of a set is well defined

$f : A \longrightarrow \underline{m}$ bijective

$g : A \longrightarrow \underline{k}$ bijective

$m \neq k$ then we can assume $m > k$

$$\underline{m} \xrightarrow[\text{bij}]{f^{-1}} A \xrightarrow[\text{bij}]{g} \underline{k}$$

injective $\lightning$

↳ cannot be true

Generalized pigeonholes principle

If $m > kn$ the for every map $f : \underline{m} \to \underline{n}$ there is a $d \in \{1, \dots, n\}$ such that

$$|f^{-1}(d)| \geq k+1$$

Proof

$$\underline{m} = \bigsqcup_{i \in \underline{n}} f^{-1}(i)$$

$j \in \underline{m}$ then

$$j \in f^{-1}(f(j))$$

$$\underline{m} = \bigcup_{i \in n} f^{-1}(i)$$

we have that

$f^{-1}(j) \cap f^{-1}(i) = \emptyset$ if $i \neq j$ if $a \in f^{-1}(j) \cap f^{-1}(i)$

then $f(u) = j$ pre image of j

$\overset{\|}{i}$ pre image of i

$\underline{m} = \bigsqcup f^{-1}(i)$

$m = |\underline{m}| = \sum_{i=1}^{n} |f^{-1}(i)|$ (rule of sum)

Assume the principle is not true $|f^{-1}(i)| \leq k$

$m \leq k \cdot n$ contradiction

$\overset{\|}{\smile}$

Example:

2 people

At a party 5 teams play a series of 1-1 games. There are two rules

- None can play twice against the same adversary
- One cannot play against their teammates.

At the end Aragon asks how many games has everyone played and get different answers.

How many games has Bilbo (Aragon 's team mate) has played?

5 teams 10 players.

Every players can play at most 8 games

Aragorn ask to 9 players how many game they have played - 9 different Answer.

Answer ∈ { 0, 1, 2 . . . 8 }

You get all the answer

0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6

Bilbo has no "partner" in the list

Bilbo played 4 games.

:)