Combinatorics Lecture 2 (3 - 4)

Slogan : Sometime it is better to "group together invariants"

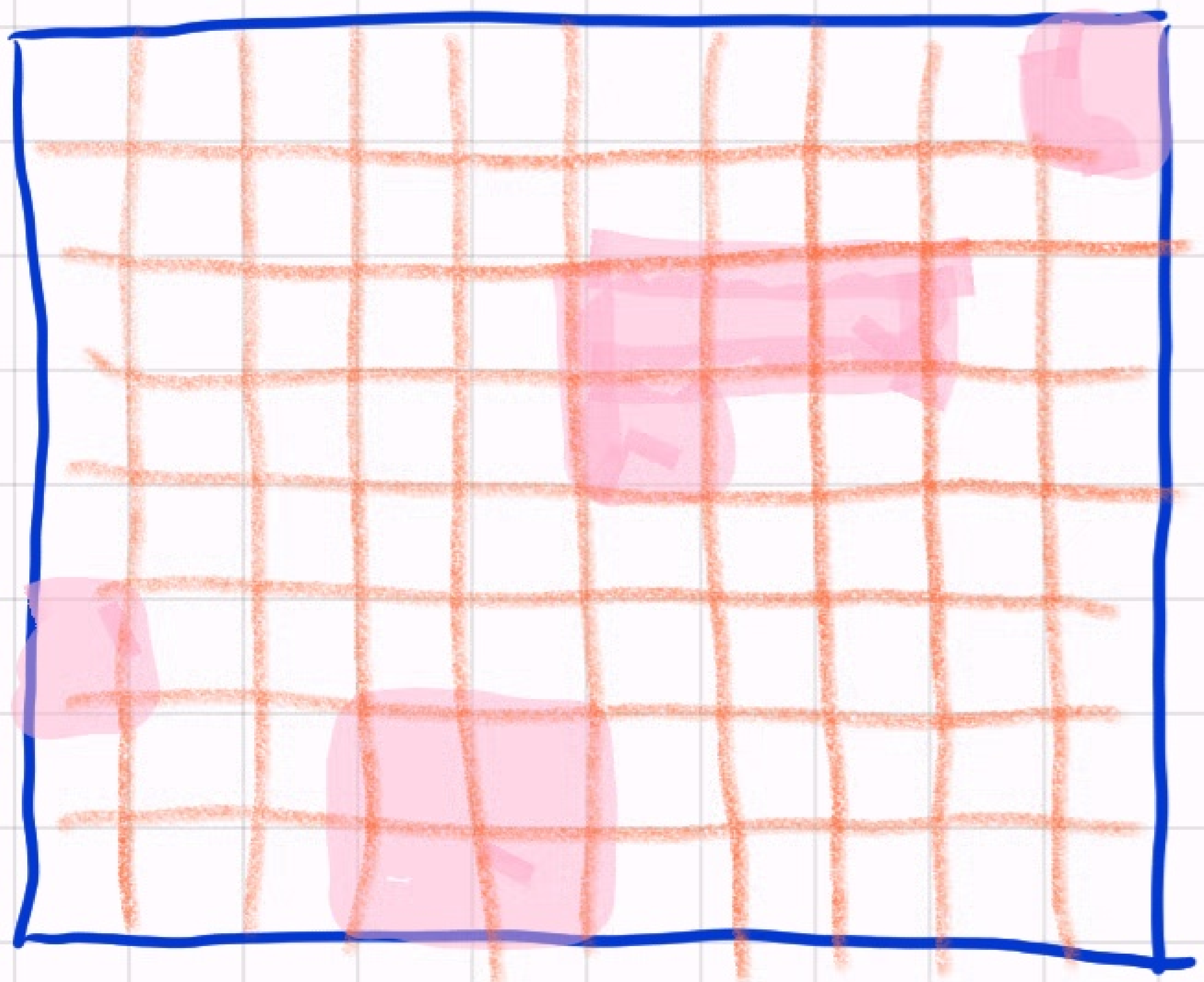
- Rook Polynomials
- Generating series

  (Partition of unity)

Rook Polynomials:

$k \in \mathbb{N}$

C - chessboard

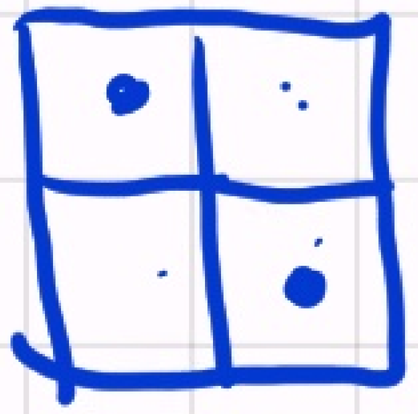


shaded $\leftrightarrow$ forbidden

$r_k(C)$ = # of ways to place k rooks () such that they do not eat eachother

(at most one in every row & every column)

Example



$r_1(C) = 4$

$r_2(C) = 2$

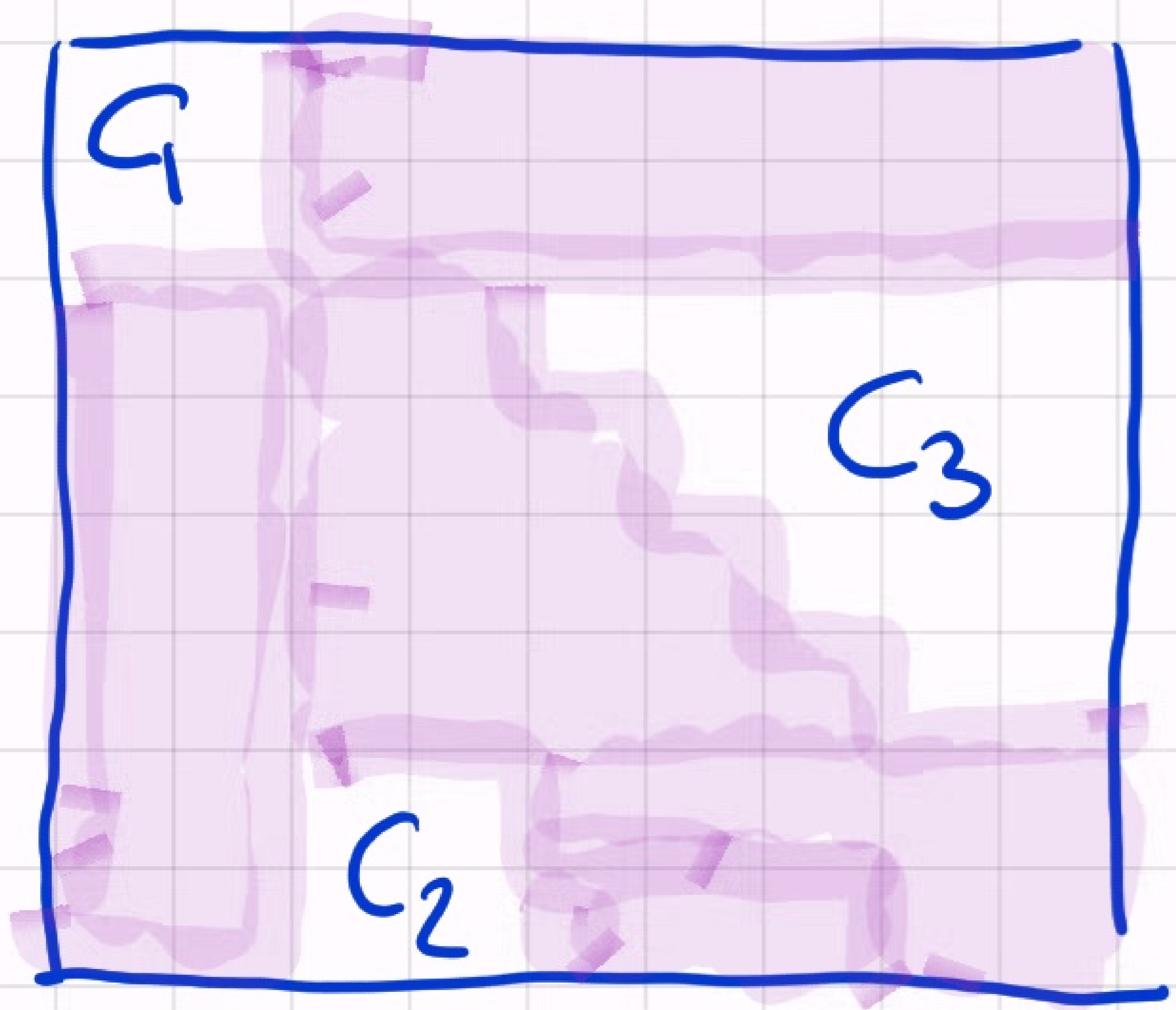
$r_3(C) = 0$

Def: The rook polynomial of a chess board C is

$$\boxed{r(C;x) = \sum_{k=0}^{\infty} r_k(C)\cdot x^k}$$

Polynomial? as k grows they will be all 0 from a certain point on

Def : Two chessboard are disjoint if they have no common rows and columns



$$C = C_1 \cup C_2 \cup C_3$$

Proposition: Let C be the (disjoint) union of $C_1 \dots C_n$ then

$$\boxed{r(C, x) = \prod_{i=1}^{n} r(C_i, x)}$$

Proof We do it for $n = 2$. The general case is with induction. $C = C_1 \cup C_2$

$$r_k(C) = \underbrace{\sum_{n=0}^{k} r_n(C_1) \cdot r_{k-n}(C_2)}_{\text{coeff of deg } k \text{ of } r(C_1,x)\cdot r(C_2,x)}$$

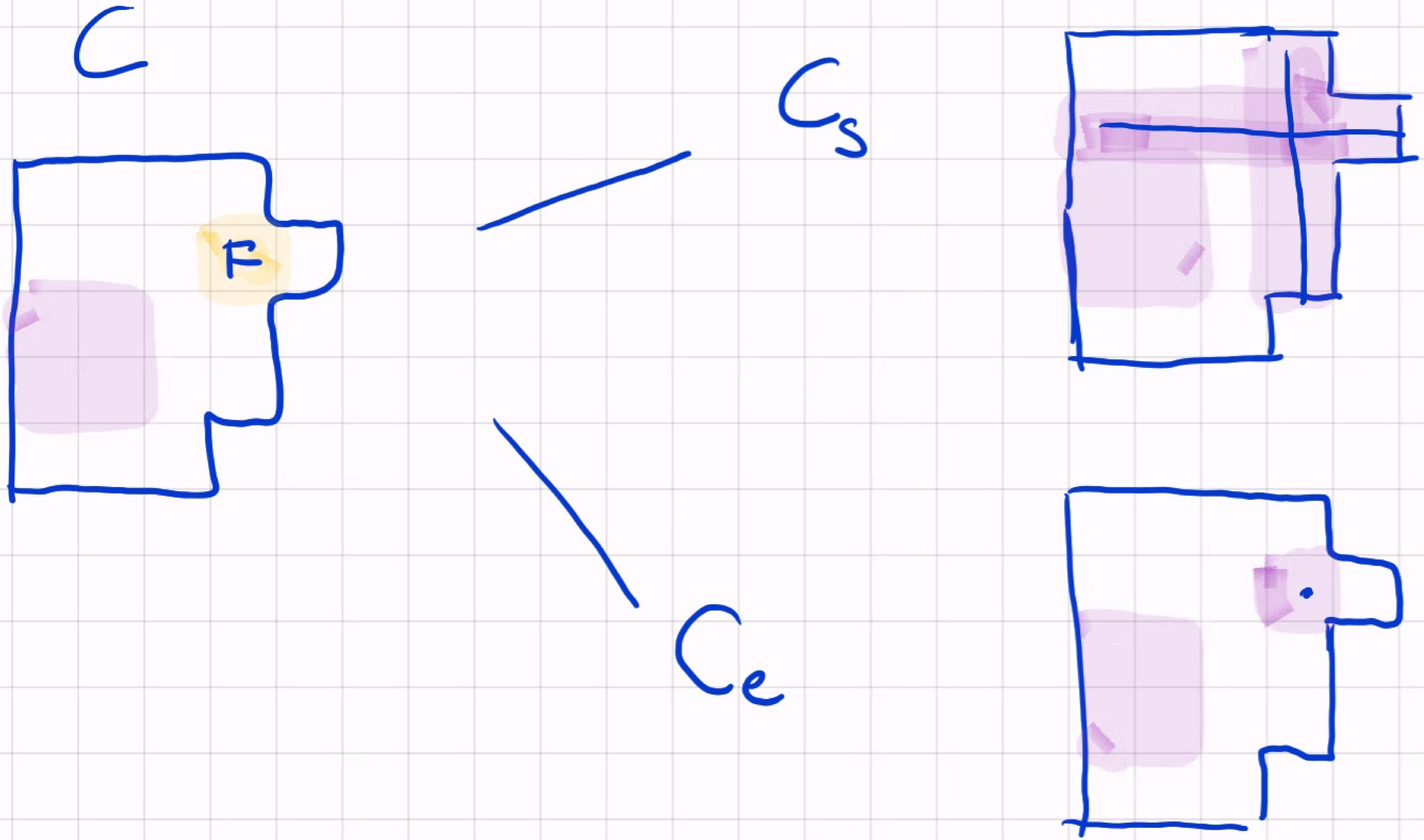
rule of product

rule of sum

Let C a chess board F a field in it

- C_s - chessboard obtained from C by eliminating the row & col from C

- C_e - chess board obtained from C by forbiding F


C
F
C_s
C_e

Proposition

$$r(C,x) = x \cdot r(C_s, x) + r(C_e, x)$$

Proof: We want to place k-rooks on C

There are two cases

① I place no rook on F $\longleftrightarrow$

\# to place k rooks in C with F forbidden

$r_k(C_e)$

② We place a rook in F

The other $k-1$ are placed in C with the row & col of F forbidden C_s

$\rightsquigarrow r_{k-1}(C_s)$

rule of sum

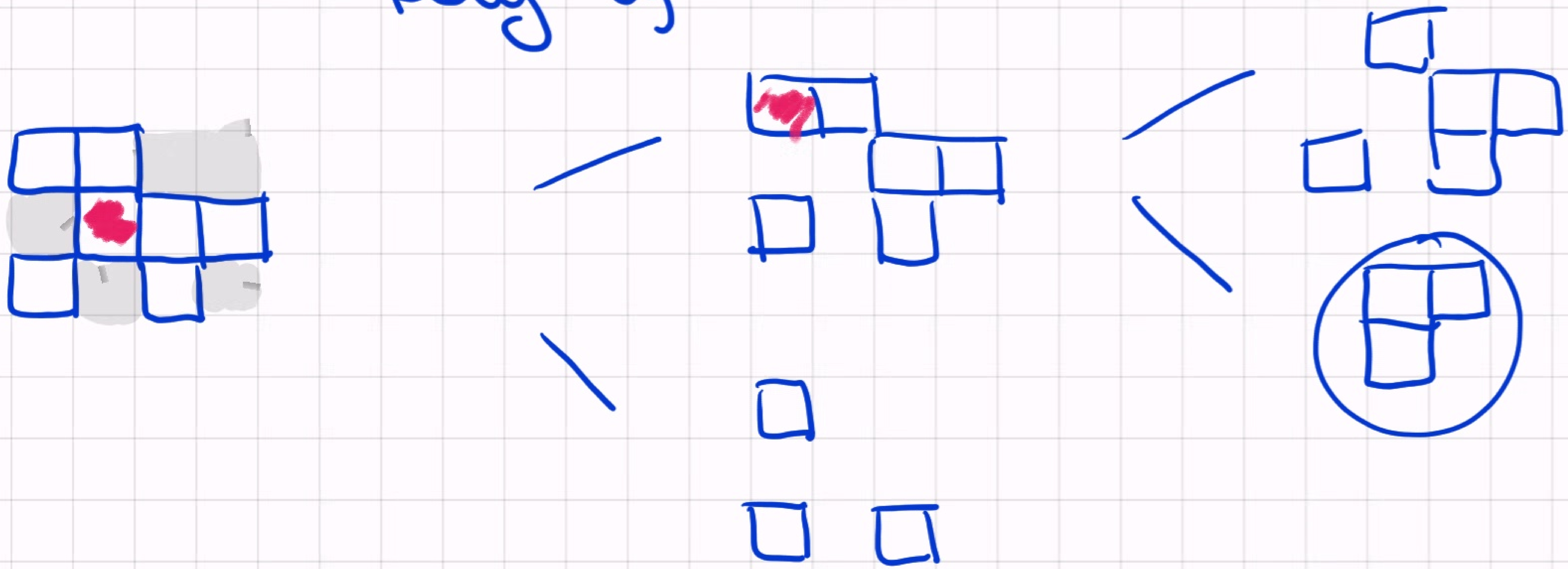
$$r_k(C) = r_{k-1}(C_s) + r_k(C_e)$$

$$r(C,x) = \underbrace{x\, r(C_s, x)} + r(C_e, x)$$

#

Example

compute the rooted poly of

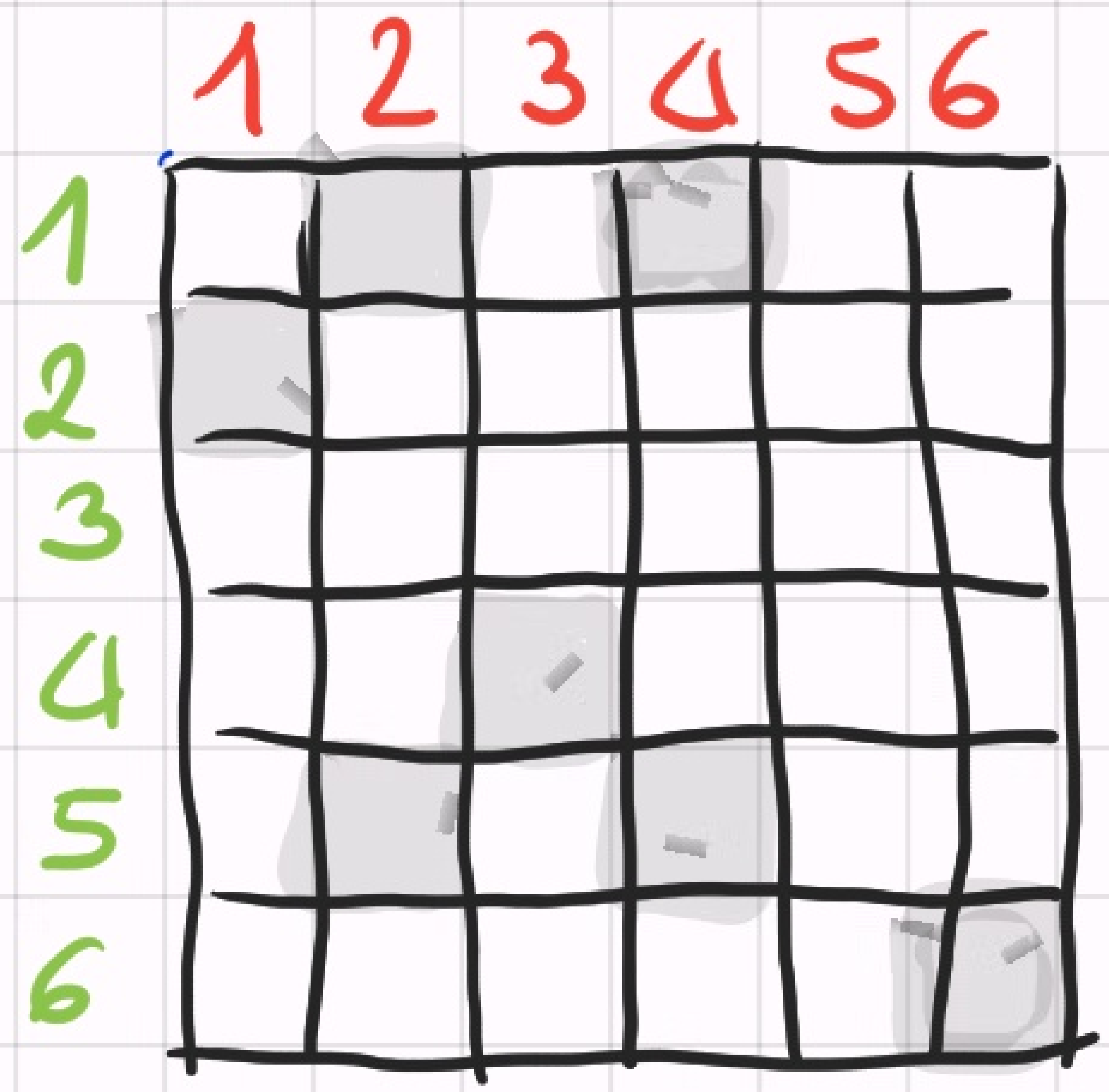


Applications

Arrangements with forbidden places

We throw two dices (a red and a green

Probability that (1,2) (2,1) (2,5) (3,4) (4,1) (4,5) (6,6) do not occur.



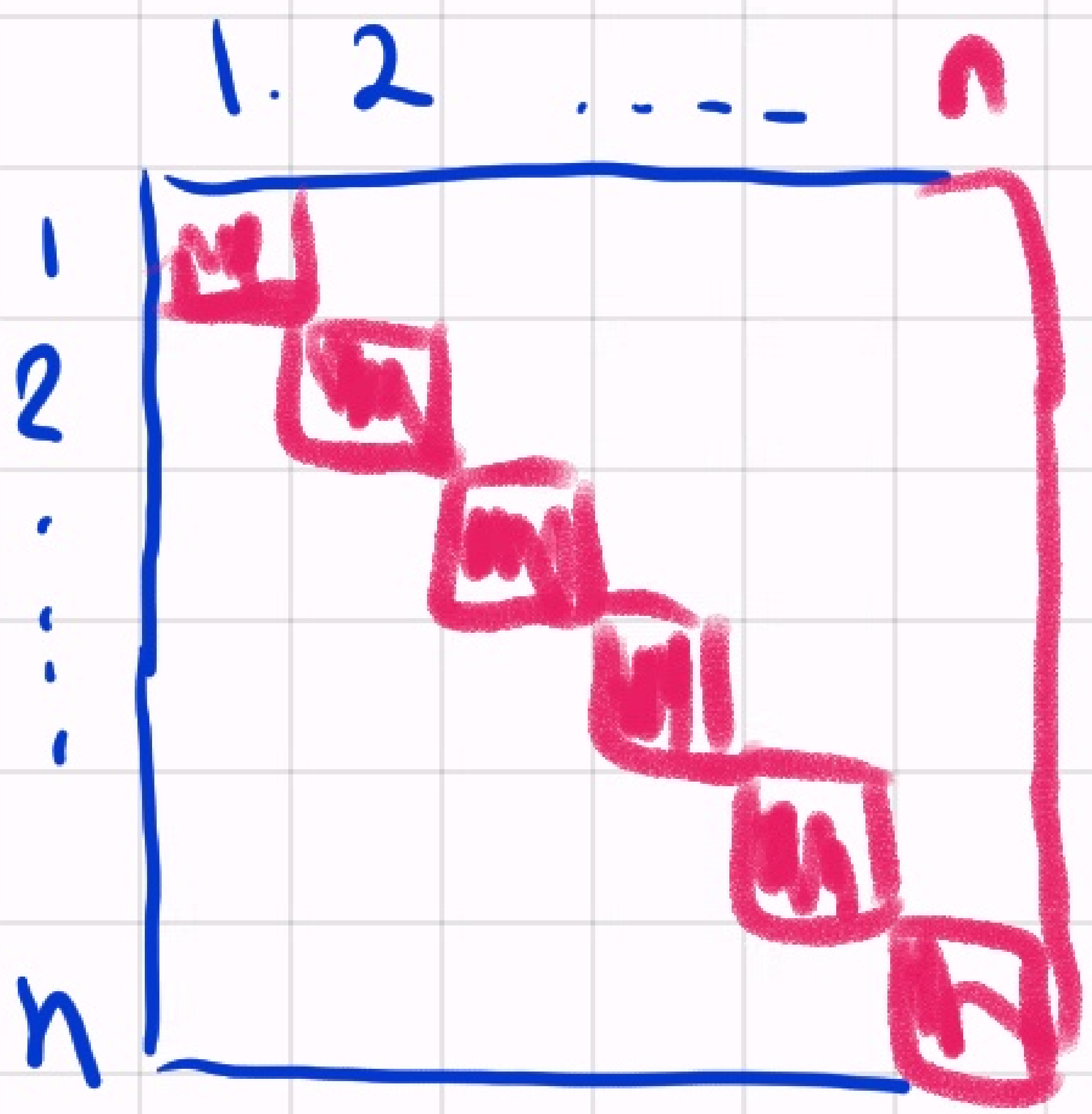
Derangements

bijective

$f: \{1 \dots n\} \to \{1 \dots n\}$

$\boxed{f(i) \neq i}$ $\forall$ $i = 1 \dots n$

$\forall \to$ for every



bijective function

1-n placy n-rooks in the table

place rook in $(i, f(i))$

$d_n = r_n(C)$

Prop A, B two finite sets

$|A| = n \leq |B| = m$ $\{B_a\}_{a \in A}$ a collection of subsets of B.

$\#$ of injective functions $f : A \longrightarrow B$ such that $f(a) \notin B_a$ for every a

$$\boxed{= \sum_{k=0}^{m} (-1)^k r_k(C, x)\, P(n-k, m-k)}$$

$$C = \begin{array}{c|ccc|} & a_1 & \cdots & a_n \\ \hline b_1 & & & \\ \vdots & & & \\ b_m & & & \\ \hline \end{array}$$

$(a\,b)$ not forbidd. $\Leftrightarrow$ $b \in B_a$

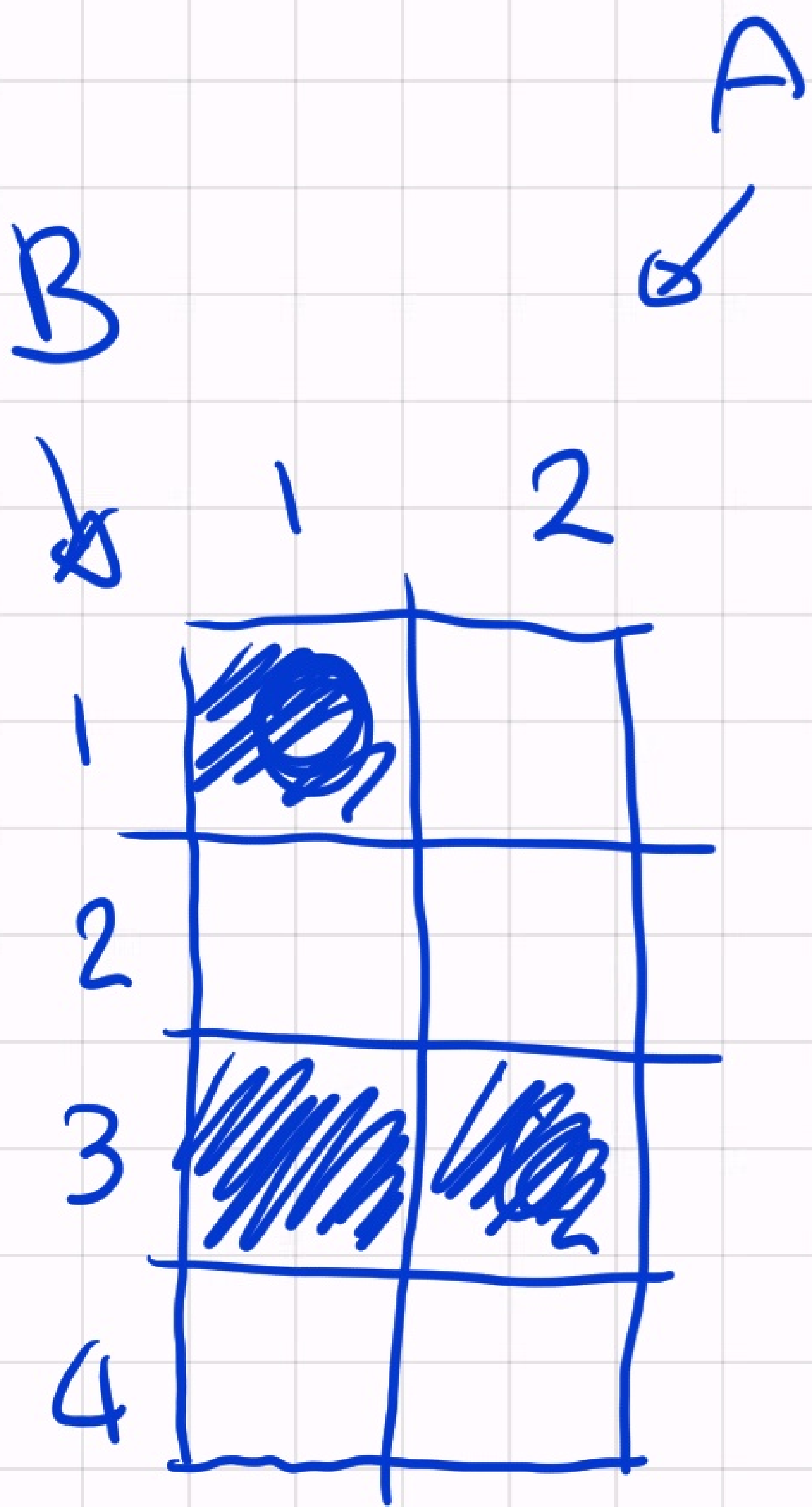
Example

$$B = \{1, 2, 3, 4\}$$

$$A = \{1, 2\}$$

$$B_1 = \{1, 3\}$$

$$B_2 = \{3\}$$



$\longleftarrow$ This is the chessboard

Proof :

$$S = \{ f: A \longrightarrow B \text{ injective} \}$$

$$c_a = \text{" } f(a) \in B_a \text{ "}$$

The number of elements of S satisfyin k conditions is

$$S_k = r_k(C)$$

I choos k colum

$$S_k = r_k(C) \cdot P(n-k, m-k)$$

$r_k(C)$ ↓ I place k rook in the chessboard

$P(n-k, m-k)$ ↓ injective funct

A \ k element ↓ B \ k element

inclusion exclusion ↓

$$N(C_a \mid a \in A) = \sum (-1)^k r_k(C) P(n-k, m-k)$$

#

Generating functions

$(a_n)_{n \in \mathbb{N}}$ a sequence. Its generating series is the formal power series

$$\sum_{n \geq 0} a_n x^n$$

We speak of generating function if this has positive convergence radius

Examples

- $r(C,x)$ is the generating function of $(r_k(C))_{k\in\mathbb{N}}$ (∞ radius of conv)

- $(1)_{n\in\mathbb{N}}$ has generating function

$$\rightsquigarrow \sum_{n=0}^{\infty} x^n \quad \left(\text{converges} \iff |x|<1\right)$$

$$= \boxed{\frac{1}{1-x}}$$

- $$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \boxed{\frac{1}{(1-x)^2}}$$

$$\frac{d}{dx}\sum_{n\geq 0} x^n = \sum_{n\geq 1} n\,x^{n-1} = \sum_{k\geq 0} (k+1)\,x^k$$

$$k = n-1$$

$$\rightsquigarrow \{1, 2, 3 \ldots\ldots\}$$

is the generating function of $(n+1)_{n\in\mathbb{N}}$

- What about $\frac{x}{(1-x)^2}$? $\quad x \cdot \frac{1}{(1-x)^2}$

$$K = n+1$$

$$x \cdot \sum_{n\geq 0}^{\infty} (n+1)\,x^n = \sum_{n=0}^{\infty} (n+1)\,x^{n+1}$$

$$\sum_{k=1}^{\infty} kx^k + 0 \cdot x^0 = \sum_{k=0}^{\infty} kx^k$$

this is the generating function of $(k)_{k \in \mathbb{N}}$

Theorem: $(a_n)_{n\in\mathbb{N}}$ $(b_n)_{n\in\mathbb{N}}$ two sequences. with generating functions f and g. Then

- $f+g$ is the generating f. of (a_n+b_n)
- $f\cdot g$ is the generating f. of

$$\Big(\underbrace{\sum_{k=0}^{n} a_k b_{n-k}}_{c_n}\Big)_{n\in\mathbb{N}}$$

Depends from how one adds & multiplies series

Applications to counting

Example 1

We want to recompute the number of ways to distribute k objects out of n

$$\boxed{\text{Answer: } \binom{n+k-1}{k}}$$

\# of object of first type

→ \# of object of 3 type

\# of ways $\quad c_1 + \cdots + c_n = k \qquad c_i \geq 0$

= coefficient of deg k of

$$\underbrace{(1+x+\cdots+x^k)\cdots\cdots(1+x+\cdots+x^k)}_{n\text{-times}}$$

$$(1+x+\quad+x^k)^n$$

$$\|$$

generating function of a problem

Coefficient of deg k of

$$\boxed{(1+x+\cdots)^n} = \left(\frac{1}{1-x}\right)^n = \frac{1}{(1-x)^n}$$

Mc Laurin expansion. $\sum_{n=0}^{\infty} \binom{n+k-1}{k} x^n$

Example : Composition of n = # ordered ways to write

$$n = n_1 + \quad + n_t \qquad n_i \geq 1$$

$2 \to$ 2

$1+1$

$3 \to$ 3

$2+1$

$1+2$

$1+1+1$

How many compositions of n with $\underline{\underline{t}}$ summands ?

$\Leftrightarrow$ # of ways in which we have

$$c_1 + \quad + c_t = n \qquad \boxed{c_i \geq 1}$$

$\Leftrightarrow$ is the coefficient of deg n of

$$\left(x + x^2 + \cdots\cdots\right)^t = \left(\frac{x}{1-x}\right)^t$$

Rule of sum

\# of decompositions

$$= \sum_{t=1}^{n} (\# \text{ of dec. with } t \text{ summands})$$

$$= \sum_{t=1}^{n} \left(\frac{x}{1-x}\right)^t$$

$$= \text{coefficient of deg } n \quad \boxed{\sum_{t=1}^{\infty} \left(\frac{x}{1-x}\right)^t}$$

$$y = \frac{x}{1-x}$$

$$\sum_{t=1}^{\infty} (y)^t = \frac{y}{1-y} = \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}} = \frac{\frac{x}{\cancel{1-x}}}{\frac{1-x-x}{\cancel{1-x}}}$$

$$= \frac{x}{1-2x} = x \cdot \sum_{m=0}^{\infty} (2x)^m$$

$$= \sum_{m=0}^{\infty} \underline{2}^m x^{m+1} \rightsquigarrow \left(2^{m-1}\right)$$

Partitions

$p(n)$ = # ways to write

$$n = m_1 + \quad + m_r \qquad m_i \geq 1$$

order irrelevant.

Examples

$p(1)$ $\quad 1 \rightsquigarrow 1$

$p(2)$ $\quad 2 = 2 = 1+1 \rightsquigarrow 2$

$p(3)$ $\quad 3 = 2+1 = 1+1+1 \rightsquigarrow 3$

$p(4)$ $\quad 4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1 \rightsquigarrow 5$

$p(5) = 7$

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1$$

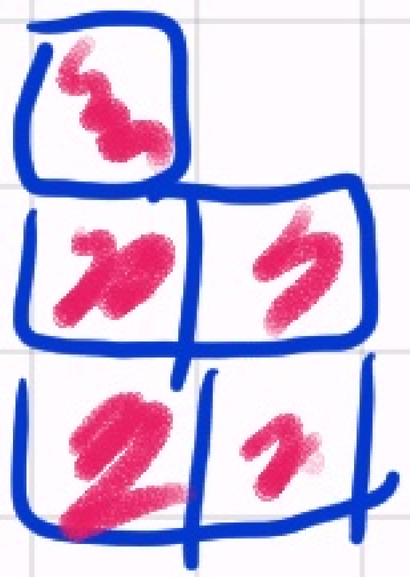
$$2+1+1+1 = 1+1+1+1+1$$

Def : Young diagram of a partition

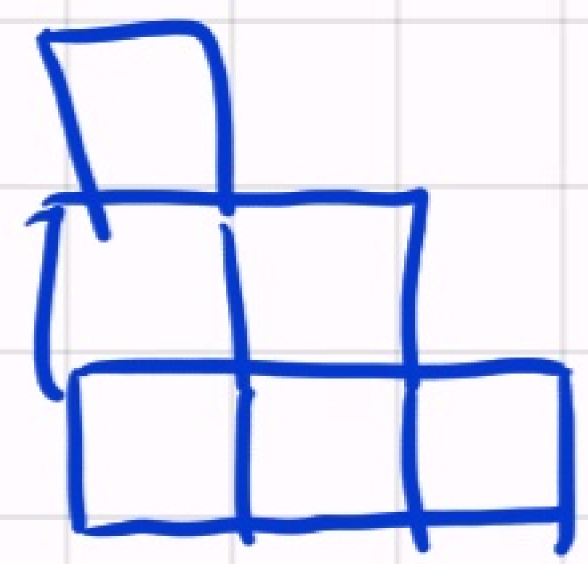
$n = n_1 + \quad + n_t \qquad n_1 \geq n_2 \geq \quad n_t$

t rows of n_i boxes

Example



$2+2+1 \qquad 5$



$3+2+1 \qquad 6$

Prop The generating function of $(p(n))_{n\in\mathbb{N}}$ is

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

Proof

$$n = \sum_k m_k \cdot k$$

how many times the summand k appears

Goal: write this as a coefficient of degree n of some series

$$(1 + x^1 + x^2 + \cdots)$$

$$m_1 = 0 \quad m_1 = 1 \quad m_2 = 2$$

$$(1 + x^2 + x^4 + \cdots)$$

$$m_2 = 0 \quad m_2 = 1 \quad m_2 = 2$$

$$(1 + x^3 + x^6 + \cdots)$$

$$\vdots$$

$$\left.\begin{matrix} (1 + x^1 + x^2 + \cdots) \\ (1 + x^2 + x^4 + \cdots) \\ (1 + x^3 + x^6 + \cdots) \\ \vdots \end{matrix}\right\} = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots$$

$$\sum_{n=0}^{\infty} p(n) x^n = \prod_{k=0}^{\infty} \frac{1}{1-x^k}$$

Observation 1 $p(n) \leq 2^{n-1} \leq 2^n$

$\downarrow$ decompositions

$$\sum p(n) x^n \leq \sum 2^n x^n$$
$$= \sum (2x)^n$$

Converges $|2x| < 1$ $|x| < \frac{1}{2}$

Positive radius of convergence.

$$\ln\left(\prod_{k=0}^{\infty}\left(\sum_{i=0}^{\infty} x^{ik}\right)\right)$$

$$= \sum_{k=0}^{\infty} \ln\left(\frac{1}{1-x^k}\right) = \sum_{k=0}^{\infty} -\ln(1-x^k)$$

$$\left|\frac{d}{dx} \cdot \ln(1-x^k)\right| = \left| + k\, x^{k-1} \frac{1}{1-x^k}\right| \leq$$

$$k c^{k-1} \frac{1}{1-c^k} \qquad \text{if } |x| < c$$

middle value theorem

$$\left| \sum \ln(1-x^k) \right| \le \sum_k k x^{k-1} \frac{1}{1-x^{k-1}} \cdot x$$

$< \infty$

converges if c small

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